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A Class of General Linear Methods for Fuzzy Differential Equations

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Abstract. This paper concerns with the developing the most general schemes so-called Fuzzy General Linear Methods (FGLM) for solving fuzzy differential equations. The general linear methods (GLM) for ordinary differential equations are the middle state of two extreme extensions (linear multistep and Runge-Kutta methods) of the one step Euler method. In this paper we study and develop the FGLMs in the case of linear multistep methods for solving fuzzy differential equations in the sense of generalized Hukuhara differentiability. The stability, consistency and convergent results will be addressed. The numerical results with a comparison is illustrated to show the efficiency and accuracy of the methods.

1. Introduction

General linear methods are a large family of numerical methods for solving ordinary differential equations. This family was invented in 1966 by Butcher [8] with the idea of extension of the simple one step Euler method. The extension of Euler method is done in two strategies: the first is to increase the steps as in the linear multistep methods and the other is to increase the complexity of one step method by taking into account more slopes as Runge-Kutta (RK) methods. The general linear methods include the middle state of these two extreme generalizations. There is an extensive study on GLMs and many authors have been developed appropriate classes for solving special differential equations

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such as the class of diagonally implicit multistage integration methods (DIMSIMs) for solving stiff problems, see e.g. [16] and references therein. On the other hand, the applications of fuzzy differential equations are developing and many problems in science and engineering are modeled well as a fuzzy differential equation.

The interpretation of a fuzzy differential equation in the sense of generalized differentiability allows to elaborately extend the extensive theory of numerical methods for ordinary differential equations to fuzzy differential equations. Many authors have been generalized the traditional methods such as Euler's method, Adams-Bashforth methods, predictor-corrector method, Runge-Kutta method[1, 2, 14, 15, 17, 20–22, 25, 28] to fuzzy differential and fuzzy initial value problems. Notion of a fuzzy derivative first introduced by Chang and Zadeh [9] and Dubo and Prade [12] introduced its extension. Stefanini [26, 27] introduced the fuzzy gH-difference and Bede and Stefanini[6] defined and studied new generalization of the differentiability for fuzzy-number-valued functions. The aim of this paper is to develop the GLMs for fuzzy differential equations and study their consistency, stability and convergence. We apply the gH-difference to interpret the fuzzy derivative[6, 26, 27].In this paper we develop a well-known Adams-Bashforth methods in the framework of a general linear method. This starting step will motivate us to develop the arbitrary classes of GLMs with demanded properties in forthcoming papers.

Let us denote by $\mathbb{R}_{\mathcal{F}}$ the class of fuzzy numbers, i.e. normal, convex, upper semicontinuous and compactly supported fuzzy subsets of the real numbers. The initial value problem for crisp and fuzzy cases is defined as follow:

$$\begin{aligned} y'(t) &= f(t, y(t)), \quad t \in [t_0, T], \\ y(t_0) &= y_0, \end{aligned} \tag{1.1}$$

where, in crisp case we have $f : \mathbb{R}^m \to \mathbb{R}^m, y_0 \in \mathbb{R}^m$ and for fuzzy variables we have $f : [t_0, T] \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ and $y_0 \in \mathbb{R}_{\mathcal{F}}$. Here, we explain the GLM for ordinary IVP (1.1) and in next sections we will discuss on the development of GLM for FIVP. Burage and Butcher [7] have presented a standard representation of a GLM in terms of four matrices. These methods were formulated as follows:

$$\begin{aligned} Y &= hAf(Y) + Uy^{[n-1]}, \\ y^{[n]} &= hBf(Y) + Vy^{[n-1]}. \end{aligned} \tag{1.2}$$

where $y^{[n-1]}$ and $y^{[n]}$ are input and output approximations, respectively, and

$$A \in \mathbb{R}^{s\times s}, \qquad U \in \mathbb{R}^{s\times r}, \qquad B \in \mathbb{R}^{r\times s}, \qquad V \in \mathbb{R}^{r\times r}.$$

In this paper we use the fuzzy interpolation for constructing Adams-Bashforth schemes in the general linear methods framework. The organization of this paper is as follow: In section 2 we present the preliminaries from GLM and fuzzy calculus. In section 3 we apply the GLM form of linear multistep methods to solve the fuzzy differential equations and in section 5 numerical results are given.

2. Preliminaries

In this section we present the required concepts from general linear methods and also we shortly review the required definitions from fuzzy calculus, as given in [3]. We will present the main idea of the paper for an important subclass of LMMs, the so-called Adams methods, in GLM framework.

A fuzzy set A is defined as a mapping $A : X \to [0,1]$ where $A(x)$ is the membership degree of x to the fuzzy set. Consider $\mathbb{R}$ as the set of all real numbers. A fuzzy subset u on $\mathbb{R}$ is called fuzzy convex if $u(tx + (1-t)y) \geq \min\{u(x), u(y)\}$, $\forall t \in [0,1]$, $x, y \in \mathbb{R}$ and is called upper semicontinuous if $\forall \epsilon > 0$ $\exists \delta > 0$ such that $u(x) - u(x_0) < \epsilon$, $|x - x_0| < \delta$ and is called normal if $\exists x_0 \in \mathbb{R}$ with $u(x_0) = 1$. The support of a fuzzy set u is defined as $\{x \in \mathbb{R}; \quad u(x) > 0\}$. A fuzzy subset u is called compactly supported if $cl\{x \in \mathbb{R}; \quad u(x) > 0\}$ is compact, where $cl(A)$ denotes the closure of the set A.

A fuzzy subset of the real line $u : \mathbb{R} \to [0,1]$ is a fuzzy number if it satisfies the following properties:

(i) u is normal;
(ii) u is fuzzy convex;
(iii) u is upper semicontinuous on $\mathbb{R}$;
(iv) u is compactly supported.

Let $\mathbb{R}_{\mathcal{F}}$ be the space of fuzzy numbers. For $0 \leq r \leq 1$, the r-level set of the fuzzy number u is defined as

$$u_r = \begin{cases} \{x \in \mathbb{R}; \quad u(x) \geq r\}, & 0 < r \leq 1, \\ cl\{x \in \mathbb{R}; \quad u(x) > 0\}, & r = 0. \end{cases}$$

Thus, if u is a fuzzy number, then $u_r = [u_r^-, u_r^+]$, where $u_r^- = \inf u_r$ and $u_r^+ = \sup u_r$ are the endpoints of the r-level set u_r. Let the functions $u^-, u^+ : [0,1] \to \mathbb{R}$ satisfy the following conditions:

(i) $u^-(r) = u_r^- \in \mathbb{R}$ is a bounded, non-decreasing, left-continuous function in $(0,1]$ and it is right-continuous at 0.
(ii) $u^+(r) = u_r^+ \in \mathbb{R}$ is a bounded, non-increasing, left-continuous function in $(0,1]$ and it is right-continuous at 0.
(iii) $u_1^- \leq u_1^+$.

Then there is a fuzzy number $u \in \mathbb{R}_{\mathcal{F}}$ that has u_r^-, u_r^+ as endpoints of its r-level sets, u_r.

A trapezoidal fuzzy number u represented by quadruple $(a, b, c, d) \in \mathbb{R}^4$ where $a \leq b \leq c \leq d$ and the endpoints of the r-level sets are given by

$$u_r = [a + r(b - a), d - r(d - c)].$$

In the representation of the trapezoidal fuzzy number if $b = c$, the fuzzy number is called a triangular fuzzy number and is given by triplet $(a, b, c) \in \mathbb{R}^3$.

Definition 2.1. Let $u, v \in \mathbb{R}_{\mathcal{F}}$, the Hukuhara difference (H-difference $\ominus_H$) of u and v is defined by

$$u \ominus_H v = w \quad \Longleftrightarrow \quad u = v + w.$$

Where $w \in \mathbb{R}_{\mathcal{F}}$ is called the H-difference of u and v. If H-difference $u \ominus_H v$ exists, then $[u \ominus_H v]_r = [u_r^- - v_r^-, u_r^+ - v_r^+]$. The Hukuhara derivative for a fuzzy function was introduced by Puri and Relescu [23].

Definition 2.2. Let $f : (a, b) \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy-number-valued function. A function f is called Hukuhara differentiable if for each $h > 0$ sufficiently close to 0, the H-differences $f(x + h) \ominus f(x)$ and $f(x) \ominus f(x + h)$ exist and

$$\lim_{h \to 0} \frac{f(x + h) \ominus f(x)}{h} = \lim_{h \to 0} \frac{f(x) \ominus f(x + h)}{h} = f'(x).$$

The fuzzy number $f'(x) \in \mathbb{R}_{\mathcal{F}}$ is called Hukuhara derivative of f at x.

Definition 2.3. The Seikkala derivative of a fuzzy function $f : (a, b) \to \mathbb{R}_{\mathcal{F}}$ is defined by

$$f'(x)_r = \left[\left(f_r^-(x)\right)', \left(f_r^+(x)\right)' \right],$$

$0 \leq r \leq 1$ and $f'(x) \in \mathbb{R}_{\mathcal{F}}$ is a fuzzy number.

From Kaleva [18] and Diamond [10], it follows that a Hukuhara differentiable function has increasing length of its support interval. So the Hukuhara difference rarely exists and to overcome this situation strongly generalized differentiability of fuzzy-number-valued functions was introduced and studied by Bede-Gal [4]. Thus, in this case a differentiable function may have the property that the support has increasing or decreasing length.

Definition 2.4. For two fuzzy numbers $u, v \in \mathbb{R}_{\mathcal{F}}$, the generalized Hukuhara difference(gH-differentiable) is the fuzzy number w, if it exists, such that

$$u \ominus_{gH} v = w \iff \begin{cases} u = v + w \\ \text{or} \\ v = u - w \end{cases}$$

If $u \ominus_{gH} v$ exists, then

$$[u \ominus_{gH} v]_r = \left[\inf_{\beta \geq r} \min \left\{ u_\beta^- - v_\beta^-, u_\beta^+ - v_\beta^+ \right\}, \sup_{\beta \geq r} \max \left\{ u_\beta^- - v_\beta^-, u_\beta^+ - v_\beta^+ \right\} \right].$$

Definition 2.5. Let $f : (a, b) \to \mathbb{R}_{\mathcal{F}}$ and $x_0 \in (a, b)$. We say that f is strongly generalized differentiable at x_0, if there exists an element $f'(x_0) \in \mathbb{R}_{\mathcal{F}}$, such that

(i) for each $h > 0$ sufficiently close to 0, the H-differences $f(x_0 + h) \ominus f(x_0)$ and $f(x_0) \ominus f(x_0 - h)$ exist and

$$\begin{aligned} \lim_{h \to 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} &= \lim_{h \to 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} \\ &= f'(x_0), \end{aligned}$$

(ii) for each $h > 0$ sufficiently close to 0, the H-differences $f(x_0) \ominus f(x_0 + h)$ and $f(x_0 - h) \ominus f(x_0)$ exist and

$$\begin{aligned} \lim_{h \to 0} \frac{f(x_0) \ominus f(x_0 + h)}{(-h)} &= \lim_{h \to 0} \frac{f(x_0 - h) \ominus f(x_0)}{(-h)} \\ &= f'(x_0). \end{aligned}$$

Let $f : (a, b) \to \mathbb{R}_{\mathcal{F}}$, we say that f is (i)-differentiable and (ii)-differentiable on (a, b) if f is differentiable in the sense (i) and (ii) of Definition 2.5, respectively.

Definition 2.6. Let $x_0 \in (a, b)$ and h be such that $x_0 + h \in (a, b)$, then the fuzzy gH-derivative of a function $f : (a, b) \to \mathbb{R}_{\mathcal{F}}$ at x_0 is defined as

$$f'_{gH}(x_0) = \lim_{h \to 0} \frac{1}{h} [f(x_0 + h) \ominus_{gH} f(x_0)]. \tag{2.1}$$

If $f'_{gH}(x_0) \in \mathbb{R}_{\mathcal{F}}$, we say that f is generalized Hukuhara differentiable (gH-differentiable) at x_0.

Theorem 2.7. *Let $f : (a, b) \to \mathbb{R}_{\mathcal{F}}$ be such that $[f(x)]_\alpha = [f_\alpha^-(x), f_\alpha^+(x)]$. Suppose that the functions $f_\alpha^-(x)$ and $f_\alpha^+(x)$ are real-valued functions, differentiable w.r.t. x, uniformly w.r.t. $\alpha \in [0, 1]$. Then the function $f(x)$ is gH-differentiable at a fixed $x \in (a, b)$ if and only if one of the following two cases holds:*

(a) $(f_\alpha^-)'(x)$ *is increasing,* $(f_\alpha^+)'(x)$ *is decreasing as functions of* α*, and* $(f_1^-)'(x) \leq (f_1^+)'(x)$*, or*
(b) $(f_\alpha^-)'(x)$ *is decreasing,* $(f_\alpha^+)'(x)$ *is increasing as functions of* α*, and* $(f_1^+)'(x) \leq (f_1^-)'(x)$*.*

Also, $\forall \alpha \in [0,1]$ *we have*

$$[f'_{gH}(x)]_\alpha = \left[\min\left\{(f_\alpha^-)'(x), (f_\alpha^+)'(x)\right\}, \max\left\{(f_\alpha^-)'(x), (f_\alpha^+)'(x)\right\}\right].$$

Proof. see[6]. □

For a fuzzy differential equations there are several interpretations. The first of them is based on the Hukuhara derivative introduced in [23] and studied by several authors [11, 18, 24, 29]. Let us consider the fuzzy initial value problem $x'(t) = f(t,x), x(t_0) = x_0 \in \mathbb{R}_\mathcal{F}$ under Hukuhara differentiability. The following lemma transforms the fuzzy differential equation into integral equations.

Lemma 2.8. *Let* $t_0 \in \mathbb{R}$*, and consider the fuzzy differential equation*

$$x'(t) = f(t,x), \qquad x(t_0) = x_0 \in \mathbb{R}_\mathcal{F},$$

where $f : \mathbb{R} \times \mathbb{R}_\mathcal{F} \longrightarrow \mathbb{R}_\mathcal{F}$ *is continuous. Then the above problem is equivalent to the integral equation*

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds,$$

on some interval $[t_0, t_1] \subset \mathbb{R}$*.*

Proof. see [3]. □

Bede in [2] proved that under certain conditions the fuzzy initial value problem

$$x'(t) = f(t,x), \qquad x(t_0) = x_0,$$

has a unique solution and is equivalent to the system of ODEs

$$\begin{cases} (x_r^-)' = f_r^-(t, x_r^-, x_r^+) \\ (x_r^+)' = f_r^+(t, x_r^-, x_r^+) \end{cases}, r \in [0,1]$$

with respect to H-differentiability.

In this interpretation solutions of a fuzzy differential equation have always an increasing length of its support interval. So a fuzzy dynamical system will have more uncertain behavior in time and it does not allow to have a periodic solutions. Thus, to solve FDEs the different ideas and methods have been investigated. The second interpretation was based on Zadeh's extension principle defined in [30]. Consider the initial value problem

$$x'(t) = f(t,x), \qquad x(0) = x_0, \tag{2.2}$$

where f is a continuous mapping from $\mathbb{R}^+ \times \mathbb{R}_{\mathcal{F}}$ into $\mathbb{R}$ and x_0 is a fuzzy number in $\mathbb{R}_{\mathcal{F}}$ with r-level intervals

$$[x_0]_r = \left[x_{0r}^-, x_{0r}^+\right], \quad 0 < r \leq 1.$$

When x is a fuzzy number the extension principle of Zadeh leads to the following definitions of $f(t,x)$:

$$f(t,x)(s) = \sup\{x(\tau) : s = f(t,\tau)\}, \quad s \in \mathbb{R}.$$

It follows that

$$[f(t,x)]_r = \left[\min\left\{f(t,u) : u \in \left[x_r^-, x_r^+\right]\right\}, \max\left\{f(t,u), u \in \left[x_r^-, x_r^+\right]\right\}\right],$$

for $x \in \mathbb{R}_{\mathcal{F}}$ with r-level sets $[x]_r = [x_r^-, x_r^+]$, $0 < r \leq 1$.

The fuzzy derivative $x'(t)$ of a fuzzy process $x : \mathbb{R}^+ \to \mathbb{R}_{\mathcal{F}}$ is defined by

$$[x'(t)]_r = \left[(x_r^-)'(t), (x_r^+)'(t)\right], \quad 0 < r \leq 1,$$

hence, $x : \mathbb{R}^+ \to \mathbb{R}_{\mathcal{F}}$ is called a fuzzy solution of (2.2) on an interval $I = [0,T)$, if

$$(x_r^-)'(t) = \min\left\{f(t,u), u \in [x_r^-(t), x_r^+(t)]\right\}, \quad x_r^-(0) = x_{0r}^-,$$
$$(x_r^+)'(t) = \max\left\{f(t,u), u \in [x_r^-(t), x_r^+(t)]\right\}, \quad x_r^+(0) = x_{0r}^+,$$

for $t \in I$ and $0 < r \leq 1$. Thus by using Zadeh's extension principle on the classical solution, we obtain a solution of the FIVP. The third interpretation have been developed based on generalized fuzzy derivative. In this work we will work with interpretation based on strongly generalized differentiability. Fuzzy differential equations based on generalized H-differentiability were investigated by Bede-Gal in [4] and more general results were proposed in Bede-Gal [5]. In Bede-Gal [4, 5] proved that under certain conditions the fuzzy initial value problem

$$x'(t) = f(t,x), \qquad x(t_0) = x_0,$$

has two unique solutions and is equivalent on some interval$[t_0, t_0 + k]$ with the union of the following two ODEs:

$$\text{(2.3)} \qquad \begin{cases} (x_r^-)'(t) = f_r^-(t, x_r^-(t), x_r^+(t)) \\ (x_r^+)'(t) = f_r^+(t, x_r^-(t), x_r^+(t)) \\ x_r^-(t_0) = (x_0)_r^-, \quad x_r^+(t_0) = (x_0)_r^+ \end{cases}, r \in [0,1]$$

$$\text{(2.4)} \qquad \begin{cases} (x_r^-)'(t) = f_r^+(t, x_r^-(t), x_r^+(t)) \\ (x_r^+)'(t) = f_r^-(t, x_r^-(t), x_r^+(t)) \\ x_r^-(t_0) = (x_0)_r^-, \quad x_r^+(t_0) = (x_0)_r^+ \end{cases}, r \in [0,1]$$

A linear multistep method is defined by the first characteristic polynomial $\rho(r) = \sum_{j=0}^{k} \alpha_j r^j$ and the second characteristic polynomial $\sigma(r) =$

$\sum_{j=0}^{k} \beta_j r^j$ as follow

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}, \tag{2.5}$$

where $a = t_n \leq t_{n+1} \leq \cdots \leq t_N = b$, $h = \frac{b-a}{N} = t_{n+k} - t_{n+k-1}$, $f_{n+j} = f(t_{n+j}, y_{n+j})$ and α_j and $\beta_j, j = 0, 1, \ldots, k$ are constants. In this scheme we can evaluate an approximate solution y_{n+k} for the exact value $y(x_{n+k})$ using the starting values $y_0, y_1, \ldots, y_{n+k-1}$. The Adams schemes are characterized by their first characteristic polynomial as $\rho(r) = r^k - r^{k-1}$. Therefor, we have

$$y_{n+k} = y_{n+k-1} + h \sum_{j=0}^{k} \beta_j f_{n+j}, \tag{2.6}$$

In (2.6) the case $\beta_k = 0$ means that the method is explicit and otherwise the method is implicit. The stability issue of LMMs are characterized by the root condition for the first characteristic polynomial $\rho(r)$, that means the roots $r_s, s = 1, 2, \ldots, k$ of $\rho(r)$ satisfy $|r_s| \leq 1$ and the roots with $|r_s| = 1$ are simple [8]. The zero-stability of an LMM and correspondingly the its GLM form depends on that the first characteristic polynomial $\rho(r)$ or the minimal polynomial of the matrix V satisfies the root condition.

3. Linear $k-$Step Methods Under Generalized H-Differentiability in the Framework of GLM

In this section, we study the GLM representation of a linear k-step method and then apply it to the numerical solution of fuzzy initial value problems. We first consider the fuzzy version of this method for solving fuzzy ordinary differential equations (FODEs) under generalized Hukuhara differentiability.

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be an equally spaced partition of the interval $[0, T]$. At the mesh point t_n, let the exact solutions under (i)- and (ii)-differentiability be denoted by

$$\mathbf{Y}_1(t_n; r) = [\mathbf{Y}_1^-(t_n; r), \mathbf{Y}_1^+(t_n; r)], \qquad \mathbf{Y}_2(t_n; r) = [\mathbf{Y}_2^-(t_n; r), \mathbf{Y}_2^+(t_n; r)],$$

respectively. Furthermore, let

$$y_1(t_n; r) = [y_1^-(t_n; r), y_1^+(t_n; r)], \qquad y_2(t_n; r) = [y_2^-(t_n; r), y_2^+(t_n; r)]$$

denote the corresponding numerical approximations at t_n under (i)- and (ii)-differentiability, respectively.

The k-step methods under generalized H-differentiability for autonomous and non-autonomous fuzzy systems under (i)-differentiability can be written as:

$$y_{1r}^{-}(t_{n+k};r) = y_{1r}^{-}(t_{n+k-1};r) + h\sum_{j=0}^{k} f^{-}(t_{n+j}, y_{1r}(t_{n+j};r)), \tag{3.1}$$

$$y_{1r}^{+}(t_{n+k};r) = y_{1r}^{+}(t_{n+k-1};r) + h\sum_{j=0}^{k} f^{+}(t_{n+j}, y_{1r}(t_{n+j};r)),$$

and under (ii)-differentiability can be written as:

$$y_{2r}^{-}(t_{n+k};r) = y_{2r}^{-}(t_{n+k-1};r) + h\sum_{j=0}^{k} f^{+}(t_{n+j}, y_{2r}(t_{n+j};r)), \tag{3.2}$$

$$y_{2r}^{+}(t_{n+k};r) = y_{2r}^{+}(t_{n+k-1};r) + h\sum_{j=0}^{k} f^{-}(t_{n+j}, y_{2r}(t_{n+j};r)),$$

A linear k-step methods (2.6) with $\rho(r) = r^k - r^{k-1}$ can be represented in the form of general linear methods. In GLM representation we should first determine the input and output vectors and then find the corresponding matrices. For this end we consider the input and output approximation of general linear methods as follow

$$y^{[n-1]} = \begin{pmatrix} y_{n+k-1} \\ hf_{n+k-1} \\ hf_{n+k-2} \\ \vdots \\ hf_{n+1} \\ hf_n \end{pmatrix}, \qquad y^{[n]} = \begin{pmatrix} y_{n+k} \\ hf_{n+k} \\ hf_{n+k-1} \\ \vdots \\ hf_{n+2} \\ hf_{n+1} \end{pmatrix}.$$

Similarly, the linear k-step methods (3.1) and (3.2) under generalized Hukuhara differentiability can be represented in the framework of general linear methods (GLMs).

For this representation, the input vectors of the GLM corresponding to (3.1) and (3.2) are denoted by

$$y_{1r}^{[n-1]} = \left[y_{1r}^{-[n-1]}, y_{1r}^{+[n-1]}\right], \qquad y_{2r}^{[n-1]} = \left[y_{2r}^{-[n-1]}, y_{2r}^{+[n-1]}\right],$$

under (i)- and (ii)-differentiability, respectively.

The corresponding output vectors are denoted by

$$y_{1r}^{[n]} = \left[y_{1r}^{-[n]}, y_{1r}^{+[n]}\right], \qquad y_{2r}^{[n]} = \left[y_{2r}^{-[n]}, y_{2r}^{+[n]}\right],$$

under (i)- and (ii)-differentiability, respectively.

We now consider the input approximation of the general linear method under generalized Hukuhara differentiability in the case of (i)-differentiability as follows:

$$y_{1r}^{-[n-1]} = \begin{pmatrix} y^{-}_{n+k-1_{1r}} \\ hf^{-}_{n+k-1_{1r}} \\ hf^{-}_{n+k-2_{1r}} \\ \vdots \\ hf^{-}_{n+1_{1r}} \\ hf^{-}_{n_{1r}} \end{pmatrix}, \qquad y_{1r}^{+[n-1]} = \begin{pmatrix} y^{+}_{n+k-1_{1r}} \\ hf^{+}_{n+k-1_{1r}} \\ hf^{+}_{n+k-2_{1r}} \\ \vdots \\ hf^{+}_{n+1_{1r}} \\ hf^{+}_{n_{1r}} \end{pmatrix}, \tag{3.3}$$

and in terms of (ii)-differentiability we obtain the following input vectors:

$$y_{2r}^{-[n-1]} = \begin{pmatrix} y^{-}_{n+k-1_{2r}} \\ hf^{+}_{n+k-1_{2r}} \\ hf^{+}_{n+k-2_{2r}} \\ \vdots \\ hf^{+}_{n+1_{2r}} \\ hf^{+}_{n_{2r}} \end{pmatrix}, \qquad y_{1r}^{+[n-1]} = \begin{pmatrix} y^{+}_{n+k-1_{2r}} \\ hf^{-}_{n+k-1_{2r}} \\ hf^{-}_{n+k-2_{2r}} \\ \vdots \\ hf^{-}_{n+1_{2r}} \\ hf^{-}_{n_{2r}} \end{pmatrix}, \tag{3.4}$$

By considering the above input vectors, the fuzzy general linear methods form of (3.1) and (3.2) can be formulated in case of (i)-differentiability as:

$$\left(\frac{Y_{1r}}{y_{1r}^{[n]}}\right) = \left(\begin{array}{c|c} \mathrm{A} & \mathrm{U} \\ \hline \mathrm{B} & \mathrm{V} \end{array}\right) \left(\frac{hf_{1r}(Y_{1r})}{y_{1r}^{[n-1]}}\right), \tag{3.5}$$

and in case of (ii)-differentiability we have:

$$\left(\frac{Y_{2r}}{y_{2r}^{[n]}}\right) = \left(\begin{array}{c|c} \mathrm{A} & \mathrm{U} \\ \hline \mathrm{B} & \mathrm{V} \end{array}\right) \left(\frac{hf_{2r}(Y_{2r})}{y_{2r}^{[n-1]}}\right), \tag{3.6}$$

where $Y_{1r} = \left[Y_{1r}^{-}, Y_{1r}^{+}\right]$ and $Y_{2r} = \left[Y_{2r}^{-}, Y_{2r}^{+}\right]$ are internal stages under (i) and (ii)-differentiability, respectively. Also, we have

$$\left(\begin{array}{c|c} \mathrm{A} & \mathrm{U} \\ \hline \mathrm{B} & \mathrm{V} \end{array}\right) = \left(\begin{array}{c|ccccc} 0 & 1 & \beta_{k-1} & \cdots & \beta_1 & \beta_0 \\ \hline 0 & 1 & \beta_{k-1} & \cdots & \beta_1 & \beta_0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{array}\right).$$

Now we present two examples of fuzzy general linear methods (FGLMs) corresponding to linear k-step methods under generalized Hukuhara differentiability for $k = 4$ and $k = 5$.

We first consider the case $k = 4$. The input vectors under (i)- and (ii)-differentiability are given, respectively, by

$$y_{1r}^{\mp[n-1]} = \begin{pmatrix} y_{1r}^{\mp}(t_{n+3}) \\ hf_{1r}^{\mp}(t_{n+3}, y_{1r}(t_{n+3})) \\ hf_{1r}^{\mp}(t_{n+2}, y_{1r}(t_{n+2})) \\ hf_{1r}^{\mp}(t_{n+1}, y_{1r}(t_{n+1})) \\ hf_{1r}^{\mp}(t_n, y_{1r}(t_n)) \end{pmatrix}, \; y_{2r}^{\mp[n-1]} = \begin{pmatrix} y_{2r}^{\mp}(t_{n+3}) \\ hf_{2r}^{\pm}(t_{n+3}, y_{2r}(t_{n+3})) \\ hf_{2r}^{\pm}(t_{n+2}, y_{2r}(t_{n+2})) \\ hf_{2r}^{\pm}(t_{n+1}, y_{2r}(t_{n+1})) \\ hf_{2r}^{\pm}(t_n, y_{2r}(t_n)) \end{pmatrix}.$$

The corresponding GLM coefficient matrix is

$$\left(\begin{array}{c|ccccc} 0 & 1 & \frac{55}{24} & \frac{-59}{24} & \frac{37}{24} & \frac{-9}{24} \\ \hline 0 & 1 & \frac{55}{24} & \frac{-59}{24} & \frac{37}{24} & \frac{-9}{24} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right).$$

Similarly, for $k = 5$, the input vectors under (i)- and (ii)-differentiability are given, respectively, by

$$y_{1r}^{\mp[n-1]} = \begin{pmatrix} y_{1r}^{\mp}(t_{n+4}) \\ hf_{1r}^{\mp}(t_{n+4}, y_{1r}(t_{n+4})) \\ hf_{1r}^{\mp}(t_{n+3}, y_{1r}(t_{n+3})) \\ hf_{1r}^{\mp}(t_{n+2}, y_{1r}(t_{n+2})) \\ hf_{1r}^{\mp}(t_{n+1}, y_{1r}(t_{n+1})) \\ hf_{1r}^{\mp}(t_n, y_{1r}(t_n)) \end{pmatrix}, \; y_{2r}^{\mp[n-1]} = \begin{pmatrix} y_{2r}^{\mp}(t_{n+4}) \\ hf_{2r}^{\pm}(t_{n+4}, y_{2r}(t_{n+4})) \\ hf_{2r}^{\pm}(t_{n+3}, y_{2r}(t_{n+3})) \\ hf_{2r}^{\pm}(t_{n+2}, y_{2r}(t_{n+2})) \\ hf_{2r}^{\pm}(t_{n+1}, y_{2r}(t_{n+1})) \\ hf_{2r}^{\pm}(t_n, y_{2r}(t_n)) \end{pmatrix}.$$

The corresponding GLM coefficient matrix is

$$\left(\begin{array}{c|cccccc} 0 & 1 & \frac{1901}{720} & \frac{-2774}{720} & \frac{2616}{720} & \frac{-1274}{720} & \frac{251}{720} \\ \hline 0 & 1 & \frac{1901}{720} & \frac{-2774}{720} & \frac{2616}{720} & \frac{-1274}{720} & \frac{251}{720} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right).$$

4. Convergence, Consistency and Stability

To investigate the convergence of the proposed FGLMs, let

$$y_1(t_{n+j}; r) = \big[y_1^-(t_{n+j}; r),\, y_1^+(t_{n+j}; r)\big]$$

and

$$y_2(t_{n+j}; r) = \big[y_2^-(t_{n+j}; r),\, y_2^+(t_{n+j}; r)\big]$$

denote the numerical solutions, and let

$$\mathbf{Y}_1(t_{n+j}; r) = \big[\mathbf{Y}_1^-(t_{n+j}; r),\, \mathbf{Y}_1^+(t_{n+j}; r)\big]$$

and

$$\mathbf{Y}_2(t_{n+j}; r) = \big[\mathbf{Y}_2^-(t_{n+j}; r),\, \mathbf{Y}_2^+(t_{n+j}; r)\big]$$

denote the corresponding exact solutions under (i)- and (ii)-differentiability, respectively. The local truncation errors (LTEs) of the FGLMs under generalized Hukuhara differentiability are defined by

$$\Psi_1(t_{n+k}; r) = \sum_{j=0}^{k} r_j y_1(t_{n+j}; r) - h\psi_{f_1}\big(y_1(t_{n+k}; r), \ldots, y_1(t_n; r)\big), \tag{4.1}$$

$$\Psi_2(t_{n+k};r) = \sum_{j=0}^{k} r_j y_2(t_{n+j};r) - h\psi_{f_2}\big(y_2(t_{n+k};r),\dots,y_2(t_n;r)\big),$$

where $r_k = -r_{k-1} = 1$ and $r_j = 0$ for $j = 0, 1, \dots, k-2$ and

$$\psi_{f_1}\big(y_1(t_{n+k};r),\dots,y_1(t_n;r)\big) = \sum_{j=0}^{k-1} \beta_j f_1(t_{n+j}, y_1(t_{n+j};r))$$

$$\psi_{f_2}\big(y_2(t_{n+k};r),\dots,y_2(t_n;r)\big) = \sum_{j=0}^{k-1} \beta_j f_2(t_{n+j}, y_2(t_{n+j};r))$$

Consistency and stability are two essential conditions for convergence.

Definition 4.1. A Fuzzy GLM form of k-step method under gH-differentiability is said to be consistent if for all fuzzy initial value problems, the residual $\Psi_1(t_{n+k};r)$ and $\Psi_2(t_{n+k};r)$ defined by (4.1) satisfies

$$\lim_{h\to 0}\frac{1}{h}\Psi_1(t_{n+k};r) = 0, \qquad \lim_{h\to 0}\frac{1}{h}\Psi_2(t_{n+k};r) = 0.$$

Definition 4.2. A Fuzzy GLM is stable if the minimal polynomial of coefficient matrix V has no zeros greater than 1 and all zeros equal to 1 are simple, in other words it satisfies the root condition.

To verify the stability of the given Fuzzy GLMs under gH-differentiability in section 3 we found the minimal polynomial $p_k(w)$ of the coefficient matrix V for $k = 4, 5$:

$$p_k(w) = w^k(w-1), \quad k = 4, 5,$$

which simply satisfies the root condition and the corresponding Fuzzy GLMs are stable.

5. Numerical Results

In this section, we solve numerically several fuzzy differential equations under generalized differentiability by FGLMs ($k = 4, 5$) presented in section 3. Also, the absolute error for each example is provided. Moreover, we presented some results concerning their convergence.Under (i)-differentiability We can estimate the order of the method's convergence by calculating the fraction $\frac{E(h/2)}{E(h)}$. It is known that $\frac{E(h/2)}{E(h)} \to O(\frac{1}{2^p})$. We calculated this fraction and provided results for each example.

Example 5.1 (Bede [3])**.** Consider the following fuzzy initial value problem

$$y' = -y + e^{-t}(-1, 0, 1), \qquad y_0 = (-1, 0, 1). \tag{5.1}$$

The system of ODEs corresponding to (i)-differentiability is given by

$$\begin{cases} (y^-)' = -y^+ - e^{-t}, \\ (y^1)' = -y^1, \\ (y^+)' = -y^- + e^{-t}, \\ y_0 = (-1, 0, 1). \end{cases}$$

The analytical solution under (i)-differentiability is

$$Y_1^-(t;r) = (1-r)\left(\frac{1}{2}e^{-t} - \frac{3}{2}e^{t}\right)$$
$$Y_1^+(t;r) = (1-r)\left(\frac{3}{2}e^{t} - \frac{1}{2}e^{-t}\right)$$

Similarly, the system of ODEs corresponding to (ii)-differentiability is given by

$$\begin{cases} (y^-)' = -y^- + e^{-t}, \\ (y^1)' = -y^1, \\ (y^+)' = -y^+ - e^{-t}, \\ y_0 = (-1, 0, 1), \end{cases}$$

and the analytical solution under (ii)-differentiability is

$$Y_2^-(t;r) = (-1+r)(1-t)\exp(-t)$$
$$Y_2^+(t;r) = (1-r)(1-t)\exp(-t).$$

We want to solve the FIVP (5.1) in the interval $[0, 2]$. The (i) and (ii)-exact and approximate solutions, resulted by FGLMs for $k = 4$ and $k = 5$, are presented in Tables 1 and 2 at $t = 2$ with $N = 20$ and $h = \frac{T-t_0}{N}$. Moreover, the results for their convergence provided in Tables 3 and 4.

From Tables 3 and 4, it follows that the Fuzzy GLMs form of 4-step methods under generalized H-differentiability turns out to have convergence order 4 and the Fuzzy GLMs form of 5-step methods under generalized H-differentiability turns out to have convergence order 5.

Example 5.2. (Khastan [19]) Consider the following linear fuzzy initial value problem:

$$\begin{aligned} y'(t) &= 2ty(t) + t(-1, 0, 1), \qquad 0 \le t \le 1, \\ y(0) &= (-1, 0, 1). \end{aligned} \tag{5.2}$$

TABLE 1. (a) Approximate solution of the FGLM ($k = 4$) $y_{1r} = [y_{1r}^-, y_{1r}^+]$, exact solution $Y_{1r} = \left[Y_{1r}^-, Y_{1r}^+\right]$ and absolute error E_{1r} under (i)-differentiability, (b) Approximate solution of the FGLM ($k = 4$) $y_{2r} = [y_{2r}^-, y_{2r}^+]$, exact solution $Y_{2r} = \left[Y_{2r}^-, Y_{2r}^+\right]$ and absolute error E_{2r} under (ii)-differentiability, Example 5.1.

r	y_{1r}	Y_{1r}	E_{1r}
0	[-1.101531E1, 1.101531E1]	[-1.101592E1, 1.101592E1]	6.024101E-4
0.1	[-9.913783E0, 9.913783E0]	[-9.914325E0, 9.914325E0]	5.421691E-4
0.2	[-8.812251E0, 8.812251E0]	[-8.812733E0, 8.812733E0]	4.819281E-4
0.3	[-7.710720E0, 7.710720E0]	[-7.711142E0, 7.711142E0]	4.216871E-4
0.4	[-6.609188E0, 6.609188E0]	[-6.609550E0, 6.609550E0]	3.614461E-4
0.5	[-5.507657E0, 5.507657E0]	[-5.507958E0, 5.507958E0]	3.012050E-4
0.6	[-4.406126E0, 4.406126E0]	[-4.406367E0, 4.406367E0]	2.409640E-4
0.7	[-3.304594E0, 3.304594E0]	[-3.304775E0, 3.304775E0]	1.807230E-4
0.8	[-2.203063E0, 2.203063E0]	[-2.203183E0, 2.203183E0]	1.204820E-4
0.9	[-1.101531E0, 1.101531E0]	[-1.101592E0, 1.101592E0]	6.024101E-5
1.0	[0,0]	[0,0]	0

(a)

r	y_{2r}	Y_{2r}	E_{2r}
0	[1.352883E-1, -1.352883E-1]	[1.353353E-1, -1.353353E-1]	4.699417E-5
0.1	[1.217595E-1, -1.217595E-1]	[1.218018E-1, -1.218018E-1]	4.229476E-5
0.2	[1.082306E-1, -1.082306E-1]	[1.082682E-1, -1.082682E-1]	3.759534E-5
0.3	[9.470180E-2, -9.470180E-2]	[9.473470E-2, -9.473470E-2]	3.289592E-5
0.4	[8.117297E-2, -8.117297E-2]	[8.120117E-2, -8.120117E-2]	2.819650E-5
0.5	[6.764414E-2, -6.764414E-2]	[6.766764E-2, -6.766764E-2]	2.349709E-5
0.6	[5.411532E-2, -5.411532E-2]	[5.413411E-2, -5.413411E-2]	1.879767E-5
0.7	[4.058649E-2, -4.058649E-2]	[4.060058E-2, -4.060058E-2]	1.409825E-5
0.8	[2.705766E-2, -2.705766E-2]	[2.706706E-2, -2.706706E-2]	9.398835E-6
0.9	[1.352883E-2, -1.352883E-2]	[1.353353E-2, -1.353353E-2]	4.699417E-6
1.0	[0,0]	[0,0]	0

(b)

The system of ODEs corresponding to (i)-differentiability is given by

$$\begin{cases} (y_{1r}^-)'(t) = 2ty_{1r}^-(t) + t(r-1), \\ (y_{1r}^+)'(t) = 2ty_{1r}^+(t) + t(1-r), \\ y_{1r}^-(0) = r-1, \quad y_{1r}^+(0) = 1-r. \end{cases}$$

The analytical solution under (i)-differentiability is

$$Y_1^-(t;r) = \frac{1}{2}\left(3e^{t^2} - 1\right)(r-1),$$
$$Y_1^+(t;r) = \frac{1}{2}\left(3e^{t^2} - 1\right)(1-r).$$

TABLE 2. (a) Approximate solution of the FGLM ($k = 5$), $y_{1r} = [y_{1r}^-, y_{1r}^+]$, exact solution $Y_{1r} = \left[Y_{1r}^-, Y_{1r}^+\right]$ and absolute error E_{1r} under (i)-differentiability, (b) Approximate solution of the FGLM ($k = 5$), $y_{2r} = [y_{2r}^-, y_{2r}^+]$, exact solution $Y_{2r} = \left[Y_{2r}^-, Y_{2r}^+\right]$ and absolute error E_{2r} under (ii)-differentiability, Example 5.1.

r	y_{1r}	Y_{1r}	E_{1r}
0	[-1.101587E1, 1.101587E1]	[-1.101592E1, 1.101592E1]	4.451187E-5
0.1	[-9.914285E0, 9.914285E0]	[-9.914325E0, 9.914325E0]	4.006069E-5
0.2	[-8.812698E0, 8.812698E0]	[-8.812733E0, 8.812733E0]	3.560950E-5
0.3	[-7.711110E0, 7.711110E0]	[-7.711142E0, 7.711142E0]	3.115831E-5
0.4	[-6.609523E0, 6.609523E0]	[-6.609550E0, 6.609550E0]	2.670712E-5
0.5	[-5.507936E0, 5.507936E0]	[-5.507958E0, 5.507958E0]	2.225594E-5
0.6	[-4.406349E0, 4.406349E0]	[-4.406367E0, 4.406367E0]	1.780475E-5
0.7	[-3.304762E0, 3.304762E0]	[-3.304775E0, 3.304775E0]	1.335356E-5
0.8	[-2.203174E0, 2.203174E0]	[-2.203183E0, 2.203183E0]	8.902375E-6
0.9	[-1.101587E0, 1.101587E0]	[-1.101592E0, 1.101592E0]	4.451187E-6
1.0	[0,0]	[0,0]	0

(a)

r	y_{2r}	Y_{2r}	E_{2r}
0	[1.353406E-1, -1.353406E-1]	[1.353353E-1, -1.353353E-1]	5.270043E-6
0.1	[1.218065E-1, -1.218065E-1]	[1.218018E-1, -1.218018E-1]	4.743039E-6
0.2	[1.082724E-1, -1.082724E-1]	[1.082682E-1, -1.082682E-1]	4.216035E-6
0.3	[9.473839E-2, -9.473839E-2]	[9.473470E-2, -9.473470E-2]	3.689030E-6
0.4	[8.120433E-2, -8.120433E-2]	[8.120117E-2, -8.120117E-2]	3.162026E-6
0.5	[6.767028E-2, -6.767028E-2]	[6.766764E-2, -6.766764E-2]	2.635022E-6
0.6	[5.413622E-2, -5.413622E-2]	[5.413411E-2, -5.413411E-2]	2.108017E-6
0.7	[4.060217E-2, -4.060217E-2]	[4.060058E-2, -4.060058E-2]	1.581013E-6
0.8	[2.706811E-2, -2.706811E-2]	[2.706706E-2, -2.706706E-2]	1.054009E-6
0.9	[1.353406E-2, -1.353406E-2]	[1.353353E-2, -1.353353E-2]	5.270043E-7
1.0	[0,0]	[0,0]	0

(b)

Similarly, the system of ODEs corresponding to (ii)-differentiability is given by

$$\begin{cases} (y_{2r}^-)'(t) = 2ty_{2r}^+(t) + t(1-r), \\ \\ (y_{2r}^+)'(t) = 2ty_{2r}^-(t) + t(r-1), \\ \\ y_{1r}^-(0) = r-1, \quad y_{1r}^+(0) = 1-r, \end{cases}$$

and the analytical solution under (ii)-differentiability is

$$Y_2^-(t;r) = \frac{1}{2}\left(3e^{-t^2} - 1\right)(r-1),$$
$$Y_2^+(t;r) = \frac{1}{2}\left(3e^{-t^2} - 1\right)(1-r).$$

TABLE 3. Convergence of the FGLM ($k = 4$)under (ii)-differentiability

r	h_i	$E_{2r}(h_i)$	p
0.2	$\frac{1}{10}$	3.759533862475462E-5	
	$\frac{1}{20}$	2.360816361970941E-6	3.993196066118324E0
	$\frac{1}{40}$	1.475860954835984E-7	3.999657112312050E0
	$\frac{1}{80}$	9.220812974275461E-9	4.000519042265663E0
0.4	$\frac{1}{10}$	2.819650396852780E-5	
	$\frac{1}{20}$	1.770612271481675E-6	3.993196066113545E0
	$\frac{1}{40}$	1.106895716057599E-7	3.999657112405316E0
	$\frac{1}{80}$	6.915609765401065E-9	4.000519034937461E0
0.6	$\frac{1}{10}$	1.879766931239119E-5	
	$\frac{1}{20}$	1.180408180992409E-6	3.993196066110909E0
	$\frac{1}{40}$	7.379304775567697E-8	3.999657112049211E0
	$\frac{1}{80}$	4.610406514893306E-9	4.000519033851667E0
0.8	$\frac{1}{10}$	9.398834656195593E-6	
	$\frac{1}{20}$	5.902040904962047E-7	3.993196066110909E0
	$\frac{1}{40}$	3.689652387783848E-8	3.999657112049211E0
	$\frac{1}{80}$	2.305203257446653E-9	4.000519033851667E0

TABLE 4. Convergence of the FGLM ($k = 5$)under (ii)-differentiability

r	h_i	$E_{2r}(h_i)$	p
0.2	$\frac{1}{10}$	4.216034534335056E-6	
	$\frac{1}{20}$	1.336319765954386E-7	4.979549509841708E0
	$\frac{1}{40}$	4.185707641601866E-9	4.996649911789974E0
	$\frac{1}{80}$	1.308334413030465E-10	4.999668298467584E0
	$\frac{1}{160}$	4.088021587911328E-12	5.000184718796247E0
0.4	$\frac{1}{10}$	3.162025900754761E-6	
	$\frac{1}{20}$	1.002239824188234E-7	4.979549510242824E0
	$\frac{1}{40}$	3.139280738140293E-9	4.996649908201587E0
	$\frac{1}{80}$	9.812499424111110E-11	4.999669576905354E0
	$\frac{1}{160}$	3.065672715685253E-12	5.000345072764134E0
0.6	$\frac{1}{10}$	2.108017267167528E-6	
	$\frac{1}{20}$	6.681598831159707E-8	4.979549509542057E0
	$\frac{1}{40}$	2.092853827739827E-9	4.996649907306344E0
	$\frac{1}{80}$	6.541663044590251E-11	4.999670292639665E0
	$\frac{1}{160}$	2.043996916167856E-12	5.000192524602108E0
0.8	$\frac{1}{10}$	1.054008633583764E-6	
	$\frac{1}{20}$	3.340799415579854E-8	4.979549509542057E0
	$\frac{1}{40}$	1.046426913869913E-9	4.996649907306344E0
	$\frac{1}{80}$	3.270831522295126E-11	4.999670292639665E0
	$\frac{1}{80}$	1.021998458083928E-12	5.000192524602108E0

By using FGLMs ($k = 4$ and $k = 5$) and taking $h = \frac{T-t_0}{N}$ where $N = 10$, the (i) and (ii)-exact and approximate solutions are presented in Tables 5 and 6 at $t = 1$. Moreover, the results for their convergence provided in Tables 7 and 8.

TABLE 5. (a)Approximate solution of the FGLM ($k = 4$) $y_{1r} = [y_{1r}^-, y_{1r}^+]$, exact solution $Y_{1r} = \left[Y_{1r}^-, Y_{1r}^+\right]$ and absolute error E_{1r} under (i)-differentiability, (b)Approximate solution of the FGLM ($k = 4$) $y_{2r} = [y_{2r}^-, y_{2r}^+]$, exact solution $Y_{2r} = \left[Y_{2r}^-, Y_{2r}^+\right]$ and absolute error E_{2r}under (ii)-differentiability, Example 5.2.

r	y_{1r}	Y_{1r}	E_{1r}
0	[-3.570017E0, 3.570017E0]	[-3.577423E0, 3.577423E0]	7.405598E-3
0.1	[-3.213015E0, 3.213015E0]	[-3.219680E0, 3.219680E0]	6.665038E-3
0.2	[-2.856014E0, 2.856014E0]	[-2.861938E0, 2.861938E0]	5.924479E-3
0.3	[-2.499012E0, 2.499012E0]	[-2.504196E0, 2.504196E0]	5.183919E-3
0.4	[-2.142010E0, 2.142010E0]	[-2.146454E0, 2.146454E0]	4.443359E-3
0.5	[-1.785009E0, 1.785009E0]	[-1.788711E0, 1.788711E0]	3.702799E-3
0.6	[-1.428007E0, 1.428007E0]	[-1.430969E0, 1.430969E0]	2.962239E-3
0.7	[-1.071005E0, 1.071005E0]	[-1.073227E0, 1.073227E0]	2.221679E-3
0.8	[-7.140034E-1, 7.140034E-1]	[-7.154845E-1, 7.154845E-1]	1.481120E-3
0.9	[-3.570017E-1, 3.570017E-1]	[-3.577423E-1, 3.577423E-1]	7.405598E-4
1.0	[0,0]	[0,0]	0

(a)

r	y_{2r}	Y_{2r}	E_{2r}
0	[-5.234721E-2, 5.234721E-2]	[-5.181916E-2, 5.181916E-2]	5.280493E-4
0.1	[-4.711249E-2, 4.711249E-2]	[-4.663725E-2, 4.663725E-2]	4.752444E-4
0.2	[-4.187777E-2, 4.187777E-2]	[-4.145533E-2, 4.145533E-2]	4.224395E-4
0.3	[-3.664305E-2, 3.664305E-2]	[-3.627341E-2, 3.627341E-2]	3.696345E-4
0.4	[-3.140833E-2, 3.140833E-2]	[-3.109150E-2, 3.109150E-2]	3.168296E-4
0.5	[-2.617361E-2, 2.617361E-2]	[-2.590958E-2, 2.590958E-2]	2.640247E-4
0.6	[-2.093888E-2, 2.093888E-2]	[-2.072766E-2, 2.072766E-2]	2.112197E-4
0.7	[-1.570416E-2, 1.570416E-2]	[-1.554575E-2, 1.554575E-2]	1.584148E-4
0.8	[-1.046944E-2, 1.046944E-2]	[-1.036383E-2, 1.036383E-2]	1.056099E-4
0.9	[-5.234721E-3, 5.234721E-3]	[-5.181916E-3, 5.181916E-3]	5.280493E-5
1.0	[0,0]	[0,0]	0

(b)

Similarly, Tables 7 and 8 show that, for Example 5.2, the fuzzy general linear methods (FGLMs) corresponding to the 4-step and 5-step methods under generalized Hukuhara differentiability achieve fourth- and fifth-order convergence, respectively.

Example 5.3 (Effati and Pakdaman [13])**.** Consider the following electrical RL circuit with an AC source:

$$y'(t) = -\frac{R}{L}y(t) + v(t), \quad t \in [0,1],$$
$$y(0) = (0.96, 1, 1.01),$$

where R is the circuit resistance and L is a coefficient corresponding to the solenoid. Consider: $R = 1$ ohm, $L = 1$ Henry and $v(t) = \sin(t)$. The system of the ODEs corresponding to (i)-differentiability is given by

$$y_{1r}^-(t) = -y_{1r}^+(t) + \sin(t),$$
$$y_{1r}^+(t) = -y_{1r}^-(t) + \sin(t),$$

TABLE 6. (a) Approximate solution of the FGLM ($k = 5$) $y_{1r} = [y_{1r}^-, y_{1r}^+]$, exact solution $Y_{1r} = \left[Y_{1r}^-, Y_{1r}^+\right]$ and absolute error E_{1r} under (i)-differentiability, (b) Approximate solution of the FGLM ($k = 5$) $y_{2r} = [y_{2r}^-, y_{2r}^+]$, exact solution $Y_{2r} = \left[Y_{2r}^-, Y_{2r}^+\right]$ and absolute error E_{2r}under (ii)-differentiability, Example 5.2.

r	y_{1r}	Y_{1r}	E_{1r}
0	[-3.575084E0, 3.575084E0]	[-3.577423E0, 3.577423E0]	2.338533E-3
0.1	[-3.217576E0, 3.217576E0]	[-3.219680E0, 3.219680E0]	2.104680E-3
0.2	[-2.860067E0, 2.860067E0]	[-2.861938E0, 2.861938E0]	1.870826E-3
0.3	[-2.502559E0, 2.502559E0]	[-2.504196E0, 2.504196E0]	1.636973E-3
0.4	[-2.145051E0, 2.145051E0]	[-2.146454E0, 2.146454E0]	1.403120E-3
0.5	[-1.787542E0, 1.787542E0]	[-1.788711E0, 1.788711E0]	1.169266E-3
0.6	[-1.430034E0, 1.430034E0]	[-1.430969E0, 1.430969E0]	9.354131E-4
0.7	[-1.072525E0, 1.072525E0]	[-1.073227E0, 1.073227E0]	7.015599E-4
0.8	[-7.150168E-1, 7.150168E-1]	[-7.154845E-1, 7.154845E-1]	4.677066E-4
0.9	[-3.575084E-1, 3.575084E-1]	[-3.577423E-1, 3.577423E-1]	2.338533E-4
1.0	[0, 0]	[0, 0]	0

(a)

r	y_{2r}	Y_{2r}	E_{2r}
0	[-5.176881E-2, 5.176881E-2]	[-5.181916E-2, 5.181916E-2]	5.035441E-5
0.1	[-4.659193E-2, 4.659193E-2]	[-4.663725E-2, 4.663725E-2]	4.531897E-5
0.2	[-4.141505E-2, 4.141505E-2]	[-4.145533E-2, 4.145533E-2]	4.028353E-5
0.3	[-3.623817E-2, 3.623817E-2]	[-3.627341E-2, 3.627341E-2]	3.524809E-5
0.4	[-3.106128E-2, 3.106128E-2]	[-3.109150E-2, 3.109150E-2]	3.021265E-5
0.5	[-2.588440E-2, 2.588440E-2]	[-2.590958E-2, 2.590958E-2]	2.517721E-5
0.6	[-2.070752E-2, 2.070752E-2]	[-2.072766E-2, 2.072766E-2]	2.014177E-5
0.7	[-1.553064E-2, 1.553064E-2]	[-1.554575E-2, 1.554575E-2]	1.510632E-5
0.8	[-1.035376E-2, 1.035376E-2]	[-1.036383E-2, 1.036383E-2]	1.007088E-5
1.0	[0, 0]	[0, 0]	0

(b)

TABLE 7. Convergence of the FGLM ($k = 4$)under (ii)-differentiability

r	h_i	$E_{2r}(h_i)$	p
0.2	$\frac{1}{10}$	4.224394786461125E-4	
	$\frac{1}{20}$	2.617571337111546E-5	4.012443905760060E0
	$\frac{1}{40}$	1.590326563376265E-6	4.040833906471029E0
	$\frac{1}{80}$	9.744893612190220E-8	4.028532798641394E0
0.4	$\frac{1}{10}$	3.168296089846243E-4	
	$\frac{1}{20}$	1.963178502842333E-5	4.012443905753868E0
	$\frac{1}{40}$	1.192744922787203E-6	4.040833906168961E0
	$\frac{1}{80}$	7.308670219030589E-8	4.028532796998010E0
0.6	$\frac{1}{10}$	2.112197393230632E-4	
	$\frac{1}{20}$	1.308785668564100E-5	4.012443905750928E0
	$\frac{1}{40}$	7.951632817297660E-7	4.040833906404671E0
	$\frac{1}{80}$	4.872446818238174E-8	4.028532795121461E0
0.8	$\frac{1}{10}$	1.056098696615316E-4	
	$\frac{1}{20}$	6.543928342820499E-6	4.012443905750928E0
	$\frac{1}{40}$	3.975816408648830E-7	4.040833906404671E0
	$\frac{1}{80}$	2.436223409119087E-8	4.028532795121461E0

TABLE 8. Convergence of the FGLM ($k = 5$)under (ii)-differentiability

r	h_i	$E_{2r}(h_i)$	p
0.2	$\frac{1}{10}$	4.028353031763909E-5	
	$\frac{1}{20}$	1.486481126536376E-6	4.760217071037054E0
	$\frac{1}{40}$	4.791373690166134E-8	4.955317998677224E0
	$\frac{1}{80}$	1.494175935756736E-9	5.003017399693968E0
0.4	$\frac{1}{10}$	3.021264773813565E-5	
	$\frac{1}{20}$	1.114860844871057E-6	4.760217071072987E0
	$\frac{1}{40}$	3.593530242124166E-8	4.955318008874479E0
	$\frac{1}{80}$	1.120631868550825E-9	5.003017496653408E0
0.6	$\frac{1}{10}$	2.014176515884036E-5	
	$\frac{1}{20}$	7.432405631710437E-7	4.760217071227110E0
	$\frac{1}{40}$	2.395686847858625E-8	4.955317996817202E0
	$\frac{1}{80}$	7.470881482896097E-10	5.003017052974895E0
0.8	$\frac{1}{10}$	1.007088257942018E-5	
	$\frac{1}{20}$	3.716202815855219E-7	4.760217071227110E0
	$\frac{1}{40}$	1.197843423929312E-8	4.955317996817202E0
	$\frac{1}{80}$	3.735440741448048E-10	5.003017052974895E0

with initial conditions

$$y_{1r}^{-}(0) = 0.96 + 0.04r, \quad y_{1r}^{+}(0) = 1.01 - 0.01r.$$

The analytic solutions under (i)-differentiability are

$$\begin{aligned} Y_1^{-}(t;r) &= \frac{1}{2}\big(\sin t - \cos t\big) + \frac{1}{2}e^{-t} \\ &\quad + (0.96 + 0.04r)\cosh t - (1.01 - 0.01r)\sinh t, \\ Y_1^{+}(t;r) &= \frac{1}{2}\big(\sin t - \cos t\big) + \frac{1}{2}e^{-t} \\ &\quad + (1.01 - 0.01r)\cosh t - (0.96 - 0.04r)\sinh t. \end{aligned}$$

Similarly, the system of the ODEs corresponding to (ii)-differentiability is given by

$$\begin{aligned} y_{2r}^{-}(t) &= -y_{2r}^{-}(t) + \sin(t), \\ y_{2r}^{+}(t) &= -y_{2r}^{+}(t) + \sin(t), \end{aligned}$$

with initial conditions

$$y_{1r}^{-}(0) = 0.96 + 0.04r, \qquad y_{1r}^{+}(0) = 1.01 - 0.01r,$$

and the analytic solutions under (ii)-differentiability are

$$\begin{aligned} Y_1^{-}(t;r) &= \frac{1}{2}(\sin(t) - \cos(t)) + \frac{1}{2}e^{-t} + (0.96 + 0.04r)e^{-t}, \\ Y_1^{+}(t;r) &= \frac{1}{2}(\sin(t) - \cos(t)) + \frac{1}{2}e^{-t} + (1.01 - 0.01r)e^{-t}. \end{aligned}$$

By applying Fuzzy GLMs presented in Section 3 ($k = 4$ and $k = 5$), for non-autonomous FODE under (i) and (ii)-differentiability, the numerical results are provided in Tables 9 and 10 at $t = 1$ with $N = 10$ and the results concerning their convergence presented in Tables 11 and 12.

TABLE 9. Approximate solutions, exact solutions, and absolute errors for the FGLMs with $k = 4$ and $k = 5$ under (i)- and (ii)-differentiability (top and bottom panels, respectively) for Example 5.3.

r	y_{1r}^{-}	Y_{1r}^{-}	E_{1r}^{-}	y_{1r}^{+}	Y_{1r}^{+}	E_{1r}^{+}
0	6.289335E-1	6.289283E-1	5.252664E-6	7.648448E-1	7.648424E-1	2.414567E-6
0.1	6.362809E-1	6.362758E-1	5.126534E-6	7.586010E-1	7.585985E-1	2.572247E-6
0.2	6.436283E-1	6.436233E-1	5.000404E-6	7.523573E-1	7.523546E-1	2.729927E-6
0.3	6.509757E-1	6.509708E-1	4.874275E-6	7.461136E-1	7.461107E-1	2.887607E-6
0.4	6.583231E-1	6.583184E-1	4.748145E-6	7.398699E-1	7.398668E-1	3.045287E-6
0.5	6.656705E-1	6.656659E-1	4.622015E-6	7.336261E-1	7.336229E-1	3.202967E-6
0.6	6.730179E-1	6.730134E-1	4.495886E-6	7.273824E-1	7.273790E-1	3.360647E-6
0.7	6.803653E-1	6.803609E-1	4.369756E-6	7.211387E-1	7.211352E-1	3.518327E-6
0.8	6.877127E-1	6.877085E-1	4.243627E-6	7.148949E-1	7.148913E-1	3.676007E-6
0.9	6.950601E-1	6.950560E-1	4.117497E-6	7.086512E-1	7.086474E-1	3.833687E-6
1.0	7.024075E-1	7.024035E-1	3.991367E-6	7.024075E-1	7.024035E-1	3.991367E-6
r	y_{2r}^{-}	Y_{2r}^{-}	E_{2r}^{-}	y_{2r}^{+}	Y_{2r}^{+}	E_{2r}^{+}
0	6.876919E-1	6.876883E-1	3.570696E-6	7.060864E-1	7.060823E-1	4.096535E-6
0.1	6.891635E-1	6.891598E-1	3.612763E-6	7.057185E-1	7.057144E-1	4.086018E-6
0.2	6.906350E-1	6.906314E-1	3.654830E-6	7.053506E-1	7.053465E-1	4.075502E-6
0.3	6.921066E-1	6.921029E-1	3.696897E-6	7.049827E-1	7.049787E-1	4.064985E-6
0.4	6.935781E-1	6.935744E-1	3.738964E-6	7.046148E-1	7.046108E-1	4.054468E-6
0.5	6.950497E-1	6.950459E-1	3.781032E-6	7.042469E-1	7.042429E-1	4.043951E-6
0.6	6.965213E-1	6.965174E-1	3.823099E-6	7.038791E-1	7.038750E-1	4.033434E-6
0.7	6.979928E-1	6.979889E-1	3.865166E-6	7.035112E-1	7.035071E-1	4.022918E-6
0.8	6.994644E-1	6.994605E-1	3.907233E-6	7.031433E-1	7.031393E-1	4.012401E-6
0.9	7.009359E-1	7.009320E-1	3.949300E-6	7.027754E-1	7.027714E-1	4.001884E-6
1.0	7.024075E-1	7.024035E-1	3.991367E-6	7.024075E-1	7.024035E-1	3.991367E-6

Similar to the above examples from Tables 11 and 12, it follows that, FGLMs ($k = 4$), and ($k = 5$) have convergence order 4 and order 5, respectively.

6. Conclusion

In this paper we have developed the linear multistep methods (Adams-Bashforth methods) in the framework of general linear methods for solving fuzzy differential equations. We have shown the consistency, stability, and convergence of the new FGLM formulation. The general framework of FGLMs will be studied in the forthcoming paper.

TABLE 10. Approximate solutions, exact solutions, and absolute errors of the FGLM with $k = 5$: (Top) under (i)-differentiability and (Below) under (ii)-differentiability for Example 5.3.

r	y_{1r}^{-}	Y_{1r}^{-}	E_{1r}^{-}	y_{1r}^{+}	Y_{1r}^{+}	E_{1r}^{+}
0	6.289268E-1	6.289283E-1	1.472007E-6	7.648407E-1	7.648424E-1	1.691358E-6
0.1	6.362743E-1	6.362758E-1	1.484306E-6	7.585968E-1	7.585985E-1	1.681723E-6
0.2	6.436218E-1	6.436233E-1	1.496606E-6	7.523529E-1	7.523546E-1	1.672087E-6
0.3	6.509693E-1	6.509708E-1	1.508906E-6	7.461090E-1	7.461107E-1	1.662452E-6
0.4	6.583168E-1	6.583184E-1	1.521206E-6	7.398652E-1	7.398668E-1	1.652816E-6
0.5	6.656643E-1	6.656659E-1	1.533505E-6	7.336213E-1	7.336229E-1	1.643181E-6
0.6	6.730119E-1	6.730134E-1	1.545805E-6	7.273774E-1	7.273790E-1	1.633546E-6
0.7	6.803594E-1	6.803609E-1	1.558105E-6	7.211335E-1	7.211352E-1	1.623910E-6
0.8	6.877069E-1	6.877085E-1	1.570405E-6	7.148897E-1	7.148913E-1	1.614275E-6
0.9	6.950544E-1	6.950560E-1	1.582704E-6	7.086458E-1	7.086474E-1	1.604639E-6
1.0	7.024019E-1	7.024035E-1	1.595004E-6	7.024019E-1	7.024035E-1	1.595004E-6
r	y_{2r}^{-}	Y_{2r}^{-}	E_{2r}^{-}	y_{2r}^{+}	Y_{2r}^{+}	E_{2r}^{+}
0	6.876868E-1	6.876883E-1	1.559479E-6	7.060807E-1	7.060823E-1	1.603885E-6
0.1	6.891583E-1	6.891598E-1	1.563032E-6	7.057128E-1	7.057144E-1	1.602997E-6
0.2	6.906298E-1	6.906314E-1	1.566584E-6	7.053449E-1	7.053465E-1	1.602109E-6
0.3	6.921013E-1	6.921029E-1	1.570137E-6	7.049771E-1	7.049787E-1	1.601221E-6
0.4	6.935728E-1	6.935744E-1	1.573689E-6	7.046092E-1	7.046108E-1	1.600333E-6
0.5	6.950443E-1	6.950459E-1	1.577242E-6	7.042413E-1	7.042429E-1	1.599445E-6
0.6	6.965158E-1	6.965174E-1	1.580794E-6	7.038734E-1	7.038750E-1	1.598556E-6
0.7	6.979874E-1	6.979889E-1	1.584347E-6	7.035055E-1	7.035071E-1	1.597668E-6
0.8	6.994589E-1	6.994605E-1	1.587899E-6	7.031377E-1	7.031393E-1	1.596780E-6
0.9	7.009304E-1	7.009320E-1	1.591452E-6	7.027698E-1	7.027714E-1	1.595892E-6
1.0	7.024019E-1	7.024035E-1	1.595004E-6	7.024019E-1	7.024035E-1	1.595004E-6

TABLE 11. Convergence of the FGLM ($k = 4$)under (ii)-differentiability

r	h_i	$E_{2r}(h_i)$	p
0.2	$\frac{1}{10}$	3.654830146082588E-6	
	$\frac{1}{20}$	2.470418029032473E-7	3.886977268573995E0
	$\frac{1}{40}$	1.588519260575794E-8	3.959000698752292E0
	$\frac{1}{80}$	1.004749727862020E-9	3.982774490881582E0
	$\frac{1}{160}$	6.314015976727205E-11	3.992134465223397E0
0.4	$\frac{1}{10}$	3.738964430843872E-6	
	$\frac{1}{20}$	2.529357326919168E-7	3.885795978084672E0
	$\frac{1}{40}$	1.627063750397895E-8	3.958428179725435E0
	$\frac{1}{80}$	1.029332841184782E-9	3.982489311507766E0
	$\frac{1}{160}$	6.469191848879063E-11	3.991980254251105E0
0.6	$\frac{1}{10}$	3.823098715938222E-6	
	$\frac{1}{20}$	2.588296625916087E-7	3.884667583631658E0
	$\frac{1}{40}$	1.665608340140068E-8	3.957881860791920E0
	$\frac{1}{80}$	1.053915732462940E-9	3.982217774038829E0
	$\frac{1}{160}$	6.624367721030922E-11	3.991832948765869E0
0.8	$\frac{1}{10}$	3.907233000477461E-6	
	$\frac{1}{20}$	2.647235919361890E-7	3.883588613084918E0
	$\frac{1}{40}$	1.704152918780011E-8	3.957360068428292E0
	$\frac{1}{80}$	1.078497180451166E-9	3.981960488815373E0
	$\frac{1}{160}$	6.779465877571056E-11	3.991706979955521E0

TABLE 12. Convergence of the FGLM ($k = 5$)under (ii)-differentiability

r	h_i	$E_{2r}(h_i)$	p
0.2	$\frac{1}{10}$	1.566584357015621E-6	
	$\frac{1}{20}$	5.843534822691510E-8	4.744637311129849E0
	$\frac{1}{40}$	1.932247806202270E-9	4.918489302930483E0
	$\frac{1}{80}$	6.179223799307465E-11	4.966710692075819E0
	$\frac{1}{160}$	1.950994921173788E-12	4.985143698514636E0
0.4	$\frac{1}{10}$	1.573689269229739E-6	
	$\frac{1}{20}$	5.870432429055938E-8	4.744540113903242E0
	$\frac{1}{40}$	1.941229399449185E-9	4.918424259602088E0
	$\frac{1}{80}$	6.208089597947719E-11	4.966677425121470E0
	$\frac{1}{160}$	1.960431816883101E-12	4.984906007877964E0
0.6	$\frac{1}{10}$	1.580794181332834E-6	
	$\frac{1}{20}$	5.897329935500295E-8	4.744443821189094E0
	$\frac{1}{40}$	1.950210104517680E-9	4.918360445086908E0
	$\frac{1}{80}$	6.237022009969451E-11	4.966628399775037E0
	$\frac{1}{160}$	1.969091556475178E-12	4.985255251457617E0
0.8	$\frac{1}{10}$	1.587899093213885E-6	
	$\frac{1}{20}$	5.924227541864724E-7	4.744348371987066E0
	$\frac{1}{40}$	1.959190920608478E-8	4.918297155403564E0
	$\frac{1}{80}$	6.265854501918966E-9	4.966602913405394E0
	$\frac{1}{160}$	1.978972541394342E-11	4.984687769602162E0

References

1. M.Z. Ahmad, M.K. Hasan and B. De Baets, *Analytical and numerical solutions of fuzzy differential equations*, Information Sciences, 236 (2013), pp. 156-167.
2. B. Bede, *Note on "Numerical solutions of fuzzy differential equations by predictor-corrector method"*, Information Sciences, 178 (2008), pp. 1917-1922.
3. B. Bede, *Mathematics of Fuzzy Sets and Fuzzy Logic*, Springer, Heidelberg, 2013.
4. B. Bede and S.G. Gal, *Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations*, Fuzzy Sets and Systems, 151 (2005), pp. 581-599.
5. B. Bede and S.G. Gal, *Solutions of fuzzy differential equations based on generalized differentiability*, Commun. Math. Anal., 9 (2010), pp. 22-41.
6. B. Bede and L. Stefanini, *Generalized differentiability of fuzzy-valued functions*, Fuzzy Sets and Systems, 230 (2013), pp. 119-141.
7. K. Burrage and J.C. Butcher, *Nonlinear stability of a general class of differential equation methods*, BIT, 20 (1980), pp. 185-203.
8. J.C. Butcher, *On the convergence of numerical solutions to ordinary differential equations*, Math. Comp., 20 (1966), pp. 1-10.
9. S.L. Chang and L.A. Zadeh, *On fuzzy mapping and control*, IEEE Trans. Systems Man Cybernet., 2 (1972), pp. 30-34.
10. P. Diamond, *Stability and periodicity in fuzzy differential equations*, IEEE Trans. Fuzzy Syst., 8 (2000), pp. 583-590.
11. Z. Ding, M. Ma and A. Kandel, *Existence of the solutions of fuzzy differential equations with parameters*, Information Sciences, 99 (1997), pp. 205-217.
12. D. Dubois and H. Prade, *Towards fuzzy differential calculus. Part 3: Differentiation*, Fuzzy Sets and Systems, 8 (1982), pp. 225-233.
13. S. Effati and M. Pakdaman, *Artificial neural network approach for solving fuzzy differential equations*, Information Sciences, 180 (2010), pp. 1434-1457.
14. M. Friedman, M. Ma and A. Kandel, *Numerical solutions of fuzzy differential and integral equations*, Fuzzy Sets and Systems, 106 (1999), pp. 35-48.
15. B. Ghazanfari and A. Shakerami, *Numerical solutions of fuzzy differential equations by extended Runge–Kutta-like formulae of order 4*, Fuzzy Sets and Systems, 189 (2012), pp. 74-91.
16. Z. Jackiewicz, *General Linear Methods for Ordinary Differential Equations*, Wiley, Hoboken, NJ, 2009.

17. O. Kaleva, *A note on fuzzy differential equations*, Nonlinear Anal., 64 (2006), pp. 895-900.
18. O. Kaleva, *Fuzzy differential equations*, Fuzzy Sets and Systems, 24 (1987), pp. 301-317.
19. A. Khastan, J.J. Nieto and R.R. Lopez, *Variation of constant formula for first order fuzzy differential equations*, Fuzzy Sets and Systems, 177 (2011), pp. 20-33.
20. H. Kim and R. Sakthivel, *Numerical solution of hybrid fuzzy differential equations using improved predictor-corrector method*, Commun. Nonlinear Sci. Numer. Simul., 17 (2012), pp. 3788-3794.
21. J.E. Macías-Díaz and S. Tomasiello, *A differential quadrature-based approach à la Picard for systems of partial differential equations associated with fuzzy differential equations*, J. Comput. Appl. Math., 297 (2016), pp. 19-38.
22. S. Ch. Palligkinis, G. Papageorgiou and I. Th. Famelis, *Runge–Kutta methods for fuzzy differential equations*, Appl. Math. Comput., 209 (2009), pp. 97-105.
23. L.M. Puri and D. Ralescu, *Differentials of fuzzy functions*, J. Math. Anal. Appl., 91 (1983), pp. 552-558.
24. R. Rodriguez-Lopez, *Comparison results for fuzzy differential equations*, Information Sciences, 178 (2008), pp. 1756-1779.
25. R.K. Saikia, *Solution of differential equation by Euler's method using fuzzy concept*, Int. J. Computer Technology and Applications, 3 (2012), pp. 226-230.
26. L. Stefanini, *A generalization of Hukuhara difference*, in *Soft Methods for Handling Variability and Imprecision*, D. Dubois, M. A. Lubiano, H. Prade, M. A. Gil, P. Grzegorzewski and O. Hryniewicz, eds., Series on Advances in Soft Computing, Vol. 48, Springer, Berlin, 2008, pp. 203-210.
27. L. Stefanini, *A generalization of Hukuhara difference and division for interval and fuzzy arithmetic*, Fuzzy Sets and Systems, 161 (2010), pp. 1564-1584.
28. S. Tapaswini and S. Chakraverty, *A new approach to fuzzy initial value problem by improved Euler method*, Fuzzy Inf. Eng., 3 (2012), pp. 293-312.
29. C. Wu, S. Song and E.S. Lee, *Approximate solutions, existence and uniqueness of the Cauchy problem of fuzzy differential equations*, J. Math. Anal. Appl., 202 (1996), pp. 629-644.
30. L.A. Zadeh, *The concept of a linguistic variable and its application to approximate reasoning*, Information Sciences, 8 (1975), pp. 199-249.

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