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An Inequality Related to Chebyshev Type for Fuzzy Integrals

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Abstract. The main goal of this paper is to establish the Chebyshev weighted-type inequality for the fuzzy integral. Several examples, both discrete and continuous, are provided to illustrate the necessity of the alignment condition for the validity of these inequalities. The results highlight the critical role of functional dependence and measure properties in determining the monotonic behavior of Sugeno integrals.

1. Introduction

The concept of fuzziness was first introduced by Zadeh [15] in 1965 and it soon found applications across various branches of science. Nearly a decade later, in 1974, Sugeno [13] extended this idea by defining the fuzzy integral, now widely known as the Sugeno integral. This integral bears a resemblance to the classical Lebesgue integral, except that the operations of addition and multiplication are replaced by the supremum and infimum, respectively.

There are two principal classes of fuzzy integrals—the Choquet and pseudo integrals—both of which are idempotent, continuous and monotonic operators. Fuzzy integral inequalities play a crucial role in many scientific and mathematical fields, including decision theory, optimization, information retrieval, differential equations, statistics and probability theory, particularly in the study of concepts such as the arithmetic and geometric means. Over the years, numerous researchers worldwide

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have explored the properties and applications of fuzzy measures and fuzzy integrals.

Ralesco and Adams [6] provided several equivalent formulations of the fuzzy integral, while Wang and Kiler [7] proposed a general framework for fuzzy measurability and integration. Later, Agahi et al. [1] established Chebyshev-type inequalities for fuzzy integrals. Building on this, Ouyang and Zhang [8] extended the main results for the Sugeno integral of monotone functions under the Lebesgue measure. Subsequently, Ouyang [9] generalized the fuzzy Chebyshev-type inequalities to arbitrary fuzzy measures. More recently, Daraby and his collaborators have derived generalized versions of Fritz–Carlson-type inequalities and other important results concerning the Sugeno integral [2–5].

Among these, the Chebyshev-type integral inequality stands out as one of the most fundamental and widely applicable inequalities in mathematics. It finds use in contexts such as the normal (and other arbitrary) distributions, especially when parameters like the mean and variance are unknown and it also plays a role in proving the weak law of large numbers. In [14], Tong stated and proved the Chebyshev-type integral inequality as follows:

Theorem 1.1 ([14])**.** *Let $f, h \in L[0,1]$ be two real-valued, continuous and comonotone functions and let μ be the Lebesgue measure on $\mathbb{R}$. Then*

$$\int_0^1 f(x)h(x)d\mu \geq \left(\int_0^1 f(x)d\mu\right)\left(\int_0^1 h(x)d\mu\right),$$

holds.

The weighted Chebyshev-type integral inequality, together with the Littlewood-type inequality, provide a foundation for the proofs of classical results such as the Grüss–Barens and Borel-type integral inequalities.

The paper is organized as follows: Section 2 introduces the basic definitions and preliminary results required for later discussions, Section 3 presents the main theorems along with illustrative examples.

2. Preliminaries

In this section, we recall some fundamental notions concerning fuzzy measures and fuzzy integrals.

Let X be a non-empty set and let Σ be a σ-algebra of subsets of X.

Definition 2.1 ([6])**.** A fuzzy measure is a set function $\mu : \Sigma \to [0, +\infty]$ satisfying the following conditions:

(i) $\mu(\emptyset) = 0$,
(ii) if $A, B \in \Sigma$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$,

(iii) if (A_n) is an increasing sequence in Σ, i.e., $A_n \subseteq A_{n+1}$ for each $n \in \mathbb{N}$, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n),$$

(iv) if (A_n) is a decreasing sequence in Σ, i.e., $A_n \supseteq A_{n+1}$ for each $n \in \mathbb{N}$ and $\mu(A_1) < \infty$, then

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

Throughout this paper, we assume that (X, Σ, μ) is a fuzzy measure space, where $\mu : \Sigma \to [0, +\infty]$ is a fuzzy measure and $\mathscr{F}_+^{\mu}(X)$ denotes the class of all non-negative, finite and μ-measurable functions with respect to Σ.

For any $f \in \mathscr{F}_+^{\mu}(X)$, we define

$$F_\alpha = \{x \in X \mid f(x) \geq \alpha\}, \quad \alpha \in [0, \infty],$$

where F_α is called the α-cut of f.

Definition 2.2 ([7])**.** Let $A \in \Sigma$ and $f \in \mathscr{F}_+^{\mu}(X)$. The fuzzy integral of f over A with respect to μ, denoted by $\oint_A f d\mu$, is defined as

$$\oint_A f d\mu = \bigvee_{\alpha\in[0,\infty]} \left[\alpha \wedge \mu(A \cap F_\alpha)\right],$$

where the symbol $\wedge$ denotes the inf operation on $[0, \infty]$.

Proposition 2.3. *Let (X, Σ, μ) be a fuzzy measure space and let $A, B \in \Sigma$ and $f \in \mathscr{F}_+^{\mu}(X)$. Then:*

(i) $\oint_A f d\mu \leq \mu(A)$;
(ii) *if $A \subseteq B$, then* $\oint_A f d\mu \leq \oint_B f d\mu$.

Definition 2.4 (function above the identity map)**.** Let $X \subseteq \mathbb{R}$ (or any totally ordered set) and let $f : X \to \mathbb{R}$ be a function. We say that f is strictly above the identity if

$$f(x) \geq x, \quad \text{for all } x \in X.$$

Definition 2.5 (function strictly above the identity map)**.** Let $X \subseteq \mathbb{R}$ (or any totally ordered set) and let $f : X \to \mathbb{R}$ be a function. We say that f is strictly above the identity if

$$f(x) > x, \quad \text{for all } x \in X.$$

Equivalently, if we define $g(x) = f(x) - x$, then f is strictly above the identity if

$$g(x) > 0, \quad \text{for all } x \in X.$$

Definition 2.6 (function below the identity map)**.** Let $X \subseteq \mathbb{R}$ (or any totally ordered set) and let $f : X \to \mathbb{R}$ be a function. We say that f is strictly below the identity if

$$f(x) \leq x, \quad \text{for all } x \in X.$$

Definition 2.7 (function strictly below the identity map)**.** Let $X \subseteq \mathbb{R}$ (or any totally ordered set) and let $f : X \to \mathbb{R}$ be a function. We say that f is strictly below the identity if

$$f(x) < x, \quad \text{for all } x \in X.$$

Equivalently, if we define $g(x) = x - f(x)$, then f is strictly below the identity if

$$g(x) > 0, \quad \text{for all } x \in X.$$

3. Main Results

Let m be a fuzzy measure on $X = [0, 1]$ and g, h be nondecreasing functions and let f be a function with $f(x) > x$ for every x. when you are considering the Sugeno integral $(S)\int$ then you are asking, which one of the following two inequalities hold? (for all admissible f, g, h, m):

$$(3.1) \qquad (S)\int_0^1 \frac{x}{f(x)} h(x)g(x)dm \leq \left[(S)\int_0^1 \frac{x}{f(x)} g(x)dm\right]\left[(S)\int_0^1 \frac{x}{f(x)} h(x)dm\right],$$

or the reversed inequality

$$(3.2) \qquad (S)\int_0^1 \frac{x}{f(x)} h(x)g(x)dm \geq \left[(S)\int_0^1 \frac{x}{f(x)} g(x)dm\right]\left[(S)\int_0^1 \frac{x}{f(x)} h(x)dm\right].$$

We will show that there is no universal inequality of either kind without extra assumptions. Concretely, we give a simple finite-space counterexample that show the first inequality does not hold and the reverse inequality can hold.

Example 3.1. Take the two–point space $X = \{1, 2\}$. Define a simple (additive) fuzzy measure by

$$m(\{1\}) = 0.6, \qquad m(\{2\}) = 0.4, \qquad m(\{1, 2\}) = 1.$$

Choose

$$f(1) = 3, \qquad f(2) = 5 \qquad (\text{so } f(x) > x),$$

and take g and h nondecreasing with

$$g(1) = h(1) = 0.2, \qquad g(2) = h(2) = 0.9.$$

Compute the three pointwise functions appearing in the integrals:

$$a(x) := f(x)g(x)h(x), \qquad b(x) := f(x)g(x), \qquad c(x) := f(x)h(x).$$

Numerical values:

x	$a(x)$	$b(x)$	$c(x)$
1	$\frac{1}{3} \times 0.2 \times 0.2 = 0.013333$	$\frac{1}{3} \times 0.2 = 0.066666$	$\frac{1}{3} \times 0.2 = 0.066666$
2	$\frac{2}{5} \times 0.9 \times 0.9 = 0.324$	$\frac{2}{5} \times 0.9 = 0.36$	$\frac{2}{5} \times 0.9 = 0.36$

Now compute Sugeno integrals (using the discrete definition):

$$(S)\int adm = \max_{t\geq 0} \min\left(t, m(\{x : a(x) \geq t\})\right).$$

For calculating a: the top value is $a(2) = 0.324$ with

$$m(\{x : a(x) \geq 0.324\}) = m(\{2\}) = 0.4.$$

Thus one candidate is $\min(0.324, 0.4) = 0.324$. The other candidate, from the smaller value, is $\min(0.01333, 1) = 0.01333$. Hence

$$(S)\int adm = 0.324.$$

For calculating b (and likewise c): the top value is 0.36 at $x = 2$ with measure 0.4, so

$$(S)\int bdm = 0.36, \qquad (S)\int cdm = 0.36.$$

Take the product on the right-hand side:

$$(S)\int bdm \cdot (S)\int cdm = 0.36 \times 0.36 = 0.1296.$$

Comparing:

$$(S)\int adm = 0.324 \quad \text{while} \quad (S)\int bdm \cdot (S)\int cdm = 0.1296,$$

We obtain,

$$\begin{aligned}(S)\int f(x)g(x)h(x)dm &= 0.324 \\ &> 0.1296 \\ &= (S)\int f(x)g(x)dm \cdot (S)\int f(x)h(x)dm.\end{aligned}$$

Hence the inequality with "$\leq$" is false in general. The reversed inequality (with "$\geq$") happens to hold in this particular example, but this do not imply that it holds universally.

Theorem 3.2. *Define*

$$u(x) := \frac{x}{f(x)}g(x), \qquad v(x) := \frac{x}{f(x)}h(x),$$

so the desired reverse inequality becomes

$$(S)\int (uv)dm \geq (S)\int udm \cdot (S)\int vdm. \tag{3.3}$$

Let

$$\alpha := (S)\int udm, \qquad \beta := (S)\int vdm.$$

If there exist levels $s, t \in [0,1]$ *such that*

$$s \leq \alpha, \qquad t \leq \beta,$$

and

$$m\left(\{x : u(x) \geq s\} \cap \{x : v(x) \geq t\}\right) \geq st, \tag{3.4}$$

then

$$(S)\int (uv)dm \geq \alpha\beta.$$

Proof. By definition of the Sugeno integral (level–set form),

$$(S)\int (uv)dm = \sup_{r\in[0,1]} \min\left(r, m(\{uv \geq r\})\right).$$

Take $r = st$. Note the elementary inclusion

$$\{x : u(x) \geq s\} \cap \{x : v(x) \geq t\} \subseteq \{x : u(x)v(x) \geq st\}.$$

Therefore, using (3.4),

$$m(\{uv \geq st\}) \geq m(\{u \geq s\} \cap \{v \geq t\}) \geq st.$$

Hence

$$\min\left(st, m(\{uv \geq st\})\right) = st.$$

Taking the supremum over r gives

$$(S)\int (uv)dm \geq st.$$

Since $s \leq \alpha$ and $t \leq \beta$, we obtain $st \leq \alpha\beta$. But we can choose s arbitrarily close to α and t arbitrarily close to β (provided (3.4) holds in the limit); in the usual cases one takes $s = \alpha$ and $t = \beta$ when the Sugeno integrals are realized as the maximizing levels. consequently,

$$(S)\int (uv)dm \geq \alpha\beta,$$

as claimed. This completes the proof that condition (3.4) is a sufficient condition for the reversed inequality (3.3). □

Remark 3.3. Condition (3.4) is often natural: it requires that the measure of the intersection of high-level sets of u and v be sufficiently large (at least equal to the product of the two level values). Intuitively, this means that the large values of the two functions must be aligned on a set of sufficiently large measure.

In particular, if u and v are highly concentrated on the same set (for example, both are large on a set A with $m(A)$ not too small), then condition (3.4) may hold.

The Lebesgue step–function example computed earlier satisfies condition (3.4) for s and t equal to the respective Sugeno levels. Indeed, in that example, the top level sets coincide with the same interval $[0.6, 1]$ and their intersection has measure 0.4. The product st computed numerically was approximately $0.310345^2 \approx 0.0963$, while the intersection measure 0.4 is larger, so condition (3.4) holds and the reverse implication follows.

In this example, the underlying space is the interval $[0, 1]$ endowed with the Lebesgue measure.

Example 3.4. Define the functions as follows:

$$f(x) \equiv 2, \qquad g(x) = h(x) = u_0(x) := \begin{cases} 0.2, & x \in [0, 0.6), \\ 0.9, & x \in [0.6, 1]. \end{cases}$$

Therefore,

$$u(x) = \frac{x}{2} u_0(x), \qquad v(x) = \frac{x}{2} u_0(x).$$

On the upper interval $x \in [0.6, 1]$, we have

$$u(x) = v(x) = 0.45x,$$

while on the lower interval $x \in [0, 0.6)$, the values of u and v are smaller than 0.27.

Computation of $\alpha = (S) \int u dm$. For $t \in [0.27, 0.45]$, the level set $\{u \geq t\}$ is given by

$$\{x \in [0.6, 1] : 0.45x \geq t\} = \{x \in [0.6, 1] : x \geq 2t/0.9\},$$

whose measure equals $1 - \frac{2t}{0.9}$. Hence, the function

$$\phi(t) := \min\left(t, 1 - \tfrac{2t}{0.9}\right)$$

attains its maximum on $[0.27, 0.45]$ at the point satisfying

$$t = 1 - \frac{2t}{0.9} \quad \Longrightarrow \quad t = \frac{1}{1 + 2/0.9} = \frac{1}{1 + 2.\overline{2}} \approx 0.310345.$$

Thus,

$$\alpha = (S) \int u dm \approx 0.310345, \qquad \beta = (S) \int v dm = \alpha.$$

Verification of condition (3.4) for $s = \alpha,\ t = \beta$. For $s = \alpha$,

$$\{u \geq \alpha\} = \{x \in [0.6, 1] : 0.45x \geq \alpha\} = \{x \in [0.6, 1] : x \geq \tfrac{\alpha}{0.45}\}.$$

Substituting $\alpha \approx 0.310345$, we get

$$\tfrac{\alpha}{0.45} \approx 0.689655, \qquad m(\{u \geq \alpha\}) = 1 - 0.689655 \approx 0.310345 = \alpha.$$

Since $u = v$, it follows that

$$m(\{u \geq \alpha\} \cap \{v \geq \beta\}) = m(\{u \geq \alpha\}) = \alpha \approx 0.310345.$$

On the other hand,

$$st = \alpha\beta = \alpha^2 \approx 0.0963.$$

Therefore, condition (3.4) is clearly satisfied because

$$m(\{u \geq \alpha\} \cap \{v \geq \beta\}) \approx 0.310345 \geq 0.0963 = \alpha^2.$$

Since the condition is satisfied, we obtain

$$(S)\int uvdm \geq \alpha\beta \approx 0.0963.$$

A direct computation gives $(S)\int uvdm = 0.288136$, which is indeed greater than $\alpha\beta$. Hence, the reverse inequality

$$(S)\int \frac{x}{f(x)}g(x)h(x)dm \geq \left((S)\int \frac{x}{f(x)}g(x)dm\right)\cdot\left((S)\int \frac{x}{f(x)}h(x)dm\right)$$

holds true in this example, with the validity grounded on the fulfillment of condition (3.4).

Remark. This example highlights the crucial role of the "overlap'' of the level sets of the functions: if the high-level regions of u and v overlap sufficiently (in the sense of (3.4)), then the Sugeno integral of the product uv attains at least the product of their individual Sugeno integrals.

Counterexample when Condition (3.4) **Fails.** : Consider the space $[0, 1]$ equipped with the Lebesgue measure and define the following two disjoint sets:

$$A = [0, 0.4], \qquad B = [0.4, 0.7].$$

Their measures are $m(A) = 0.4,\ m(B) = 0.3$ and clearly $A \cap B = \varnothing$. Define the functions u and v as the indicator functions of these sets:

$$u(x) = \mathbf{1}_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A, \end{cases} \qquad v(x) = \mathbf{1}_B(x) = \begin{cases} 1, & x \in B, \\ 0, & x \notin B. \end{cases}$$

In this case, we have

$$u(x)v(x) = \mathbf{1}_{A\cap B}(x) \equiv 0, \quad (\text{since } A \cap B = \varnothing).$$

Now we compute the Sugeno integrals. For an indicator function $\mathbf{1}_E$ with $m(E) = \gamma \in [0,1]$, we have

$$(S)\int \mathbf{1}_E dm = \sup_{t\in[0,1]} \min\big(t, m(\{\mathbf{1}_E \geq t\})\big).$$

For $t \in (0,1]$, the level set $\{\mathbf{1}_E \geq t\} = E$ and hence

$$\min\big(t, m(\{\mathbf{1}_E \geq t\})\big) = \min(t,\gamma).$$

This function attains its maximum at $t = \gamma$, so

$$(S)\int \mathbf{1}_E dm = \gamma.$$

Accordingly, we obtain:

$$(S)\int u dm = m(A) = 0.4, \qquad (S)\int v dm = m(B) = 0.3.$$

Thus, their product is

$$(S)\int u dm \cdot (S)\int v dm = 0.4 \times 0.3 = 0.12.$$

On the other hand, since $uv \equiv 0$,

$$(S)\int (uv) dm = (S)\int 0 dm = 0.$$

Therefore, in this simple example,

$$(S)\int (uv) dm = 0 < 0.12 = (S)\int u dm \cdot (S)\int v dm,$$

meaning that the reverse inequality does not hold.

Verification of Condition (3.4)**.** If we take $s = (S)\int u dm = 0.4$ and $t = (S)\int v dm = 0.3$ in this example, then

$$m(\{u \geq s\} \cap \{v \geq t\}) = m(A \cap B) = 0,$$

while

$$st = 0.12.$$

Hence, the condition

$$m(\{u \geq s\} \cap \{v \geq t\}) \geq st$$

is violated, which explains why the reverse inequality fails.

4. Conclusion

In this study, we investigated a general form of the Chebyshev-type inequality for the Sugeno integral defined on fuzzy measure spaces and abstract spaces. As a direction for future research, it would be of interest to extend and examine analogous Chebyshev-type inequalities for other types of fuzzy integrals, such as the Choquet, pseudo and quantum integrals.

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