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A New Framework for Non-Newtonian Conformable Fractional Cauchy Problems

Mohammad Jahanshahi[1] and Reza Danaei[2*]

Abstract. In this paper, we define multiplicative time-fractional differential equations using conformable derivatives. We then introduce the Laplace transform of multiplicative conformable fractional differential equations and derive the multiplicative Laplace transforms of the specified equations and elementary functions. Finally, we demonstrate the effectiveness of the proposed method by presenting some examples and graphs of the solutions.

1. Introduction

Fractional calculus is an extension of classical calculus to non-integer orders and has emerged as a powerful mathematical tool for modeling complex systems characterized by memory and hereditary properties. Fractional derivatives capture the complex behaviors of phenomena such as viscoelasticity, anomalous diffusion, and biological processes, which are often inadequately represented by classical models compared to standard integer-order derivatives. Several definitions of fractional derivatives exist, including the Grünwald-Letnikov, Riemann-Liouville, Caputo, Caputo-Fabrizio, and Atangana-Baleanu formulations. Each of these definitions has its advantages depending on the specific application. Conformable fractional derivatives have garnered significant interest due to their simplicity, intuitive definition, and compatibility

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* Corresponding author.

with classical calculus principles [1, 20]. hese characteristics make conformable derivatives particularly appealing for both theoretical investigation and practical application, Numerical and analytical-approximate solution methods have been developed for various types of these equations.[2, 3, 9, 10, 12, 14, 16–18]. Time-fractional differential equations have recently emerged as fundamental in the study of systems that exhibit time-dependent memory phenomena. Multiplicative time-fractional equations, a specific subset of these equations, appear in a variety of scientific and engineering applications, including control theory, signal processing, economics, mathematical modeling, and heat transfer [4, 5, 7, 8, 13, 22, 24, 28, 29]. For instance, in biology and ecology, they are used to model population dynamics. In economics, multiplicative models describe economic growth and are also utilized in financial modeling. In the field of chemical kinetics, Multiplicative fractional equations are applied to enzyme kinetics and catalytic processes. Furthermore, in industrial automation, multiplicative control strategies regulate processes where the control action is proportional to the product of system states, also relevant in communication systems, where they model multiplicative noise and interference. In renewable energy systems, such as solar panels and wind turbines, the efficiency of energy conversion is often a multiplicative function of environmental factors. Additionally, this equations are employed in portfolio optimization and risk management [11, 21, 23, 24, 26, 28, 29]. However, analytically solving such equations remains a significant challenge due to their inherent complexity and the limitations of current solution techniques. The Laplace transform, a well-known method for solving and investigating differential equations, offers a promising approach to addressing these challenges. Its ability to convert differential equations into algebraic forms simplifies the solution process, making it an essential tool in both classical and fractional calculus [6, 15, 25, 27, 30].

In this paper, we provide a comprehensive explanation of the mathematical framework, derive the Laplace transform for multiplicative time-fractional equations using the conformable derivative, and demonstrate the practicality of the proposed method through examples and graphical evaluations. This investigation not only advances the theoretical understanding of fractional calculus but also offers practical tools for tackling complex challenges in science and engineering.

2. Preliminaries

In this section we give some about conformable fractional derivative and miltiplicative fractional calculus, [1, 8, 13, 20, 24, 30].

2.1. **Conformable Fractional Derivative.**

Definition 2.1. Given a function $\varphi : [0,\infty) \longrightarrow \mathbb{R}$. Then the conformable fractional derivative of φ of order α, $0 < \alpha \leq 1$ is defined by

$$D^{\alpha}(\varphi)(t) = \lim_{\epsilon \to 0} \frac{\varphi(t + \epsilon t^{1-\alpha}) - \varphi(t)}{\epsilon}$$

for all $t > 0$.

The relationship between the conformable derivative and the first derivative can be represented

$$D^{\alpha}(\varphi)(t) = t^{1-\alpha}\varphi'(t).$$

We can see that D^{α} satisfies in the following theorem.

Theorem 2.2. *Let $0 < \alpha < 1$ and φ, ψ be α-differentiable at a point $t > 0$, then we have*

(i) $D^{\alpha}(a\varphi + b\psi) = aD^{\alpha}(f) + bD^{\alpha}(g)$, *for all* $a, b \in \mathbb{R}$,
(ii) $D^{\alpha}(t^{p}) = pt^{p-\alpha}$, *for all* $p \in \mathbb{R}$,
(iii) $D^{\alpha}(c) = 0$, *for all constant functions* $\varphi(t) = c$,
(iv) $D^{\alpha}(\varphi\psi) = \varphi D^{\alpha}(\psi) + \psi D^{\alpha}(\varphi)$,
(v) $D^{\alpha}(\frac{\varphi}{\psi}) = \frac{\psi D^{\alpha}(\varphi) + \varphi D^{\alpha}(\psi)}{\psi^{2}}$.

Proof. See [20]. □

Definition 2.3. Let $0 < \alpha \leq 1$ and $\varphi : [a,b] \longrightarrow \mathbb{R}$ be continous function, then

$$\begin{aligned} J^{\alpha}(\varphi)(t) &= \int_{0}^{t} x^{\alpha-1}\varphi(x)dx \\ &= \int_{0}^{t} \varphi(x)d_{\alpha}x, \end{aligned}$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0,1]$.

Lemma 2.4. *Let $\alpha \in (0,1]$ and φ be a continuous differentiable function on $[a,\infty)$, then we have*

$$J_{a}^{\alpha} D_{a}^{\alpha}\varphi(t) = \varphi(t) - \varphi(a), \quad \forall t > a.$$

Definition 2.5. The conformable fractional exponantial function for every $t \geq a$, is defined by

$$\begin{aligned} E_{\alpha}(\zeta, t-a) &= \exp\left[\zeta \frac{(t-a)^{\alpha}}{\alpha}\right] \\ &= \sum_{k=0}^{\infty} \frac{\zeta^{k}(t-a)^{\alpha k}}{\alpha^{k}k!}, \quad 0 < \alpha \leq 1, \quad \zeta \in \mathbb{R}. \end{aligned}$$

Now, we will present some fundamental definitions and properties of conformable fractional Laplace transform which can be found in [1, 30].

Definition 2.6. Let $\varphi : [0,\infty] \longrightarrow \mathbb{R}$ be a real valued function and $0 < \alpha \leq 1$, the the conformable fractional Laplace transform of φ for $x \geq 0$ is defined by

$$\begin{aligned} \mathcal{L}_\alpha\{\varphi(t)\}(s) &= \Phi_\alpha(s) \\ &= \int_0^\infty x^{\alpha-1} E_\alpha(-s,t)\varphi(x)dx \\ &= \int_0^\infty E_\alpha(-s,t)\varphi(x)d_\alpha x. \end{aligned} \tag{2.1}$$

Theorem 2.7. *Let $\varphi : [0,\infty] \longrightarrow \mathbb{R}$ be differentiable real-valued function and $0 < \alpha \leq 1$, then*

$$\mathcal{L}_\alpha\left\{D^\alpha\varphi(t)\right\}(s) = s\Phi_\alpha(s) - \varphi(0).$$

Theorem 2.8. *Let $\varphi : [0,\infty) \longrightarrow \mathbb{R}$ be a cantinuous real-valued differentiable function and $0 < \alpha \leq 1$, then*

$$\mathcal{L}_\alpha\left\{D^{2\alpha}\varphi(t)\right\}(s) = s^2\Phi_\alpha(s) - s\varphi(0) - D^\alpha\varphi(0).$$

Lemma 2.9. *Let $\varphi : [0,\infty] \longrightarrow \mathbb{R}$ be a function such that $L_\alpha\{\varphi(t)\}(s) = \Phi_\alpha(s)$ exists. Then*

$$\Phi_\alpha(s) = \mathcal{L}\{\varphi(\alpha t)^{\frac{1}{\alpha}}\}(s),$$

where $\mathcal{L}\{\phi(t)\}(s) = \int_0^\infty e^{-st}\phi(t)dt$.

2.2. **Multiplicative Calculus.**

Definition 2.10. Let $\varphi : \mathbb{R} \to \mathbb{R}^+$ be a positive function. The multiplicative derivative of the function φ is given by

$$\frac{d^*\varphi}{dt}(t) = \varphi^*(t) = \lim_{h\to 0}\left(\frac{\varphi(t+h)}{\varphi(t)}\right)^{\frac{1}{h}}.$$

We assum that φ is a positive function and using properties of the classical derivative, then multiplicative derivative can be written as

$$\frac{d^*\varphi(t)}{dt} = \varphi^*(t) = e^{\frac{\varphi'(t)}{\varphi(t)}} = e^{(\ln\varphi(t))'}. \tag{2.2}$$

Theorem 2.11. *Suppose φ be a positive and differentiable function that is multiplicative differentiable at the point t, then it is differentiable in the classical sense, and we have*

$$\varphi'(t) = \varphi(t)\ln\varphi^*(t).$$

Theorem 2.12. *Let φ and ψ be differentiable with the multiplicative derivative. If c is an arbitrary constant, then the functions $c\cdot\varphi$, $\varphi\cdot\psi$, $\varphi+\psi$, φ/ψ, φ^ψ have multiplicative derivatives given by*

(i) $(c \cdot \varphi)^*(t) = \varphi^*(t)$,
(ii) $(\varphi \cdot \psi)^*(t) = \varphi^*(t) \cdot \psi^*(t)$,
(iii) $(\varphi + \psi)^*(t) = \varphi^*(t)^{\frac{\varphi(t)}{\varphi(t)+\psi(t)}} + \psi^*(t)^{\frac{\psi(t)}{\varphi(t)+\psi(t)}}$,
(iv) $(\varphi/\psi)^*(t) = \varphi^*(t)/\psi^*(t)$,
(v) $(\varphi^\psi)^*(t) = \varphi^*(t)^{\psi(t)}\varphi(t)^{\psi'(t)}$.

Proposition 2.13. *Suppose that $\varphi(t)$ is a positive function, then we have, $\varphi^*(t) = 1$ if and only if $\varphi'(t) = 0$.*

Definition 2.14. If φ is positive and Riemann integrable on $[a, b]$, then the multiplicative integral of φ is as follows

$$\int_a^b \varphi(t)^{dt} = \exp\left(\int_a^b \ln \varphi(t)dt\right) = e^{\int_a^b \ln \varphi(t)dt}.$$

We have the following properties for this multiplicative integral:

(i) $\int_a^b (\varphi(t)^k)^{dt} = \left(\int_a^b \varphi(t)^{dt}\right)^k$, $k \in \mathbb{R}$,
(ii) $\int_a^b (\varphi(t) \cdot \psi(t))^{dt} = \int_a^b \varphi(t)^{dt} \cdot \int_a^b \psi(t)^{dt}$,
(iii) $\int_a^b \left(\frac{\varphi(t)}{\psi(t)}\right)^{dt} = \frac{\int_a^b \varphi(t)^{dt}}{\int_a^b \psi(t)^{dt}}$,
(iv) $\int_a^b \varphi(t)^{dt} = \int_a^c \varphi(t)^{dt} \cdot \int_c^b \varphi(t)^{dt}$, $a \le c \le b$,

where φ and ψ are multiplicative integrable functions on $[a, b]$.

2.3. **Multiplicative conformable fractional derivative.** In this part we define the multiplicative conformable fractional derivative of order $\alpha \in (0, 1]$ for a positive real-valued function φ defined on $[0, a]$.

Definition 2.15. Let $\varphi : \mathbb{R} \to \mathbb{R}^+$ be a positive function and $\alpha \in (0, 1]$, then multiplicative conformable fractional derivative of the function φ is given by

$$({}^*D^\alpha \varphi)(t) = \lim_{\zeta \to 0} \left(\frac{\varphi\left(t + \zeta t^{1-\alpha}\right)}{\varphi(t)}\right)^{\frac{1}{\zeta}}.$$

Now by using (2.2), we have

$$\begin{aligned} ({}^*D^\alpha \varphi)(t) &= \exp\left(D^\alpha \ln(\varphi(t))\right) \\ &= \exp\left(\frac{D^\alpha \varphi(t)}{\varphi(t)}\right) \\ &= \exp\left(\frac{t^{1-\alpha}\varphi'(t)}{\varphi(t)}\right) \end{aligned} \tag{2.3}$$

Definition 2.16. Let $\alpha \in (0, 1]$ and $\varphi : \mathbb{R} \to \mathbb{R}^+$ be a positive function. The multiplicative conformable fractional integral of the function φ is

given by

$$\begin{aligned}(^*J^\alpha\varphi)(t) &= \int_0^a \varphi(t)^{\left(x^{\alpha-1}\right)^{dt}} \\ &= \exp\left(\int_0^a x^{\alpha-1}\ln\varphi(t)dt\right) \\ &= e^{\int_0^a x^{\alpha-1}\ln\varphi(t)dt},\end{aligned}$$

or in this form

$$\begin{aligned}(^*J^\alpha\varphi)(t) &= \int_0^a \varphi(t)^{d_\alpha t} \\ &= \exp\left(\int_0^a \ln\varphi(t)d_\alpha t\right) \\ &= e^{\int_0^a \ln\varphi(t)d_\alpha t},\end{aligned}$$

Lemma 2.17. *Let φ be a real-valued function defined on $[0,a]$ and $\alpha\in(0,1]$, then*

(i) $(^*D_\alpha\ {}^*J_\alpha\varphi)(t)=\varphi(t)$, *for φ is continuous.*
(ii) $(^*J_\alpha\ {}^*D_\alpha\ \varphi)(t)=\frac{\varphi(t)}{\varphi(a)}$.

In the following, we define the Laplace transform method for multiplicative fractional differential equations that utilize conformable fractional derivatives. We will also apply this method to investigate and solve relevant problems of interest.

3. Analysis of the Method

In this section, we define a new Laplace transform rule for solving multiplicative fractional differential equations with conformable derivatives, utilizing the classical Laplace transform. We derive the properties and characteristics of the relevant functions and apply this rule to solve the corresponding equations.

Definition 3.1. Let $\varphi(t)$ be a positive definite function defined on the closed interval $[0,\infty)$, then the multiplicative Laplace transform of $\varphi(t)$ is defined as

$$\begin{aligned}\mathcal{L}_m\{\varphi(t)\} &= \Phi_m(s) \\ &= \int_0^\infty \varphi(t)^{E_\alpha(-s,t)^{d_\alpha t}} \\ &= e^{\int_0^\infty \ln\varphi(t)E_\alpha(-s,t)d_\alpha t} \\ &= e^{\mathcal{L}\{\ln\varphi(t)\}},\end{aligned}$$

or

$$\mathcal{L}_m\{\varphi(t)\} = \exp(\mathcal{L}\{\ln\varphi(t)\}) = \Phi_m(s).$$

The multiplicative Laplace transforms of elementary functions are as follows:

(i) $\mathcal{L}_m\{1\} = \exp(\mathcal{L}\{\ln 1\}) = 1$,

(ii) $\mathcal{L}_m\left\{e^{\frac{t^\alpha}{\alpha}}\right\} = \exp\left(\mathcal{L}\left\{\ln\frac{t^\alpha}{\alpha}\right\}\right) = \exp\left(\frac{1}{s}\right)$,

(iii) $\mathcal{L}_m\left\{e^{e^{\frac{at^\alpha}{\alpha}}}\right\} = \exp\left(\mathcal{L}\left\{\ln e^{\frac{at^\alpha}{\alpha}}\right\}\right) = \exp(\frac{1}{s-a})$,

(iv) $\mathcal{L}_m\left\{e^{\sin\frac{at^\alpha}{\alpha}}\right\} = \exp\left(\mathcal{L}\left\{\ln\sin\frac{at^\alpha}{\alpha}\right\}\right) = \exp\left(\frac{a}{s^2+a^2}\right)$,

(v) $\mathcal{L}_m\left\{e^{\cos\frac{at^\alpha}{\alpha}}\right\} = \exp\left(\mathcal{L}\left\{\ln\cos\frac{at^\alpha}{\alpha}\right\}\right) = \exp\left(\frac{s}{s^2+a^2}\right)$,

(vi) $\mathcal{L}_m\left\{e^{\sinh\frac{at^\alpha}{\alpha}}\right\} = \exp\left(\mathcal{L}\left\{\ln\sinh\frac{at^\alpha}{\alpha}\right\}\right) = \exp\left(\frac{a}{s^2-a^2}\right)$,

(vii) $\mathcal{L}_m\left\{e^{\cosh\frac{at^\alpha}{\alpha}}\right\} = \exp\left(\mathcal{L}\left\{\ln\cosh\frac{at^\alpha}{\alpha}\right\}\right) = \exp\left(\frac{s}{s^2-a^2}\right)$.

These relationships can be obtained using (2.1).

Theorem 3.2. *If c_1, c_2 are arbitary constants and φ_1, φ_2 are two functions that possess multiplicative Laplace transform, then the following holds*

$$\mathcal{L}_m\left\{\varphi_1^{c_1}\varphi_2^{c_2}\right\} = \mathcal{L}_m\{\varphi_1\}^{c_1}\mathcal{L}_m\{\varphi_2\}^{c_2}.$$

This indicates that the multiplicative Laplace transform has a linear property.

Theorem 3.3. *If $\mathcal{L}_m\{\varphi(t)\} = \Phi_m(s)$, then*

$$\mathcal{L}_m\left\{\varphi(t)^{e^{a\frac{t^\alpha}{\alpha}}}\right\} = \Phi_m(s-a).$$

Proof. See [28]. □

Theorem 3.4. *Let $\mathcal{L}_m\{\varphi(t)\} = \Phi_m(s)$. Then we have*

$$\mathcal{L}_m\left\{\varphi\left(a\frac{t^\alpha}{\alpha}\right)\right\} = \Phi_m\left(\frac{s}{a}\right)^{\frac{1}{a}}.$$

Proof. We can write

$$\mathcal{L}_m\left\{\varphi\left(a\frac{t^\alpha}{\alpha}\right)\right\} = \exp\left\{\int_0^\infty \ln\varphi\left(a\frac{t^\alpha}{\alpha}\right)E_\alpha(-s,t)d_\alpha t\right\}.$$

Now, by using the substitution $u = a\frac{t^\alpha}{\alpha}$ in the right-hande side integral above, we have $\frac{1}{a}du = t^{\alpha-1}dt = d_\alpha t$, we obtain

$$\mathcal{L}_m\left\{\varphi\left(a\frac{t^\alpha}{\alpha}\right)\right\} = \exp\left\{\frac{1}{a}\int_0^\infty \ln f(u)E_\alpha(-s,t)dt\right\}$$

$$= \exp\left\{\int_0^\infty (\ln f(u)E_\alpha(-s,t))^{\frac{1}{a}}\, dt\right\}$$
$$= \Phi_m\left(\frac{s}{a}\right)^{\frac{1}{a}}. \qquad \square$$

Theorem 3.5. *Let $\mathcal{L}_m\{\varphi(t)\} = \Phi_m(s)$. Then we have*

$$\mathcal{L}_m\left\{\varphi\left(\frac{t^\alpha}{\alpha}\right)^{\frac{t^\alpha}{\alpha}}\right\} = \left(\frac{d^*}{ds}\Phi_m(s)\right)^{-1}.$$

Proof. By using multiplicative Laplace transform, we have

$$\mathcal{L}_m\{\varphi\left(\frac{t^\alpha}{\alpha}\right)\} = \exp\left\{\int_0^\infty \ln\varphi\left(\frac{t^\alpha}{\alpha}\right) E_\alpha(-s,t)d_\alpha t\right\}.$$

If we take the multiplicative fractional derivative of this relation, we will obtain

$$\begin{aligned}
{}^*\Phi_m^\alpha(s) &= \frac{d^{*\alpha}}{d_\alpha s}\left(\exp\left\{\int_0^\infty \ln\varphi\left(\frac{t^\alpha}{\alpha}\right) E_\alpha(-s,t)d_\alpha t\right\}\right)\\
&= \exp\left[\frac{d^\alpha}{ds^\alpha}\left(\exp\left\{\int_0^\infty \ln\varphi\left(\frac{t^\alpha}{\alpha}\right) E_\alpha(-s,t)d_\alpha t)\right\}\right)\right]\\
&= \exp\left[-\int_0^\infty \frac{t^\alpha}{\alpha}\ln\varphi\left(\frac{t^\alpha}{\alpha}\right) E_\alpha(-s,t)d_\alpha t\right]\\
&= \left(\exp\left[\int_0^\infty \ln\varphi\left(\frac{t^\alpha}{\alpha}\right)^{\frac{t^\alpha}{\alpha}} E_\alpha(-s,t)d_\alpha t\right]\right)^{-1}.
\end{aligned}$$

This completes the proof. $\square$

Definition 3.6. Let $\varphi(t)$ be a non-negative function that definite on $[0,\infty)$, and $\ln\varphi(t)$ be a function on $[0,\infty)$. If there exist positive constants t_0, C and γ such that

$$|\varphi(t)| \le C\exp(E_\alpha(\gamma,t)),$$

for $t > t_0$, then φ function is said to be of γ-double exponential order.

Theorem 3.7. *Let $\varphi(t)$ be a non-negative function that is of γ-double exponential order for $t > t_0$ on $[0,\infty)$, and $\ln\varphi(t)$ be piecewise continuous function on $[0,\infty)$. Then for $s > \gamma$, $\mathcal{L}_m\varphi(t)$ exists.*

Proof. See [28]. $\square$

Theorem 3.8. *Let $\varphi(t)$ be a function of γ-double exponential order for $t > t_0$ on $[0,A]$. Let ${}^*D^\alpha\varphi$ be piecewise continuous function given*

on $[0, A]$. *Then the Laplace transform of multiplicative conformable fractional derivative for* $s > \gamma$ *is*

$$\mathcal{L}_m\{{}^*D^\alpha(\varphi(t)\} = \frac{\Phi(s)^s}{\varphi(0)}, \quad \alpha \in (0, 1].$$

Proof. Using Definition (2.3), we have

$$\begin{aligned}\mathcal{L}_m\{{}^*D^\alpha(\varphi(t))\} &= \mathcal{L}_m\left\{\exp\left(\frac{t^{1-\alpha}\varphi'(t)}{\varphi(t)}\right)\right\} \\ &= \exp\left(\int_0^\infty \frac{t^{1-\alpha}\varphi'(t)}{\varphi(t)} E_\alpha(-s,t)d_\alpha t\right) \\ &= \exp\left(\lim_{C\longrightarrow\infty}\int_0^C \frac{t^{1-\alpha}\varphi'(t)}{\varphi(t)} E_\alpha(-s,t)d_\alpha t\right).\end{aligned}$$

Now using the integration by parts method, we have

$$\begin{aligned}&\int_0^C \frac{t^{1-\alpha}\varphi'(t)}{\varphi(t)} E_\alpha(-s,t)d_\alpha t \\ &= \int_0^C \frac{\varphi'(t)}{\varphi(t)} E_\alpha(-s,t)dt \\ &= \lim_{C\longrightarrow\infty} E_\alpha(-s,t)\ln\varphi(t)\Big|_0^C + \lim_{C\longrightarrow\infty}\int_0^C s\ln\varphi(t)E_\alpha(-s,t)dt \\ &= -\ln\varphi(0) + s\mathcal{L}\{\ln\varphi(t)\}.\end{aligned}$$

So we obtain

$$\begin{aligned}\mathcal{L}_m\{{}^*D^\alpha\varphi(t)\} &= \exp\left(-\ln\varphi(0) + s\mathcal{L}\{\ln\varphi(t)\}\right) \\ &= \frac{\Phi(s)^s}{\varphi(0)}.\end{aligned}$$

This completes the proof. □

We derive the following corollary by utilizing the properties of the conformable fractional derivative [2, 3].

Corollary 3.9. *Let* $\varphi : [0, \infty) \longrightarrow \mathbb{R}$ *is conformable differentiable of order* $n (n \in \mathbb{N})$ *a, then we obtain Laplace transform of multiplicative conformable fractional derivative* ${}^*D^{2\alpha}\varphi(t)$ *for* $s > \alpha$ *as follows*

$$\mathcal{L}_m\left\{{}^*D^{2\alpha}\varphi(t)\right\} = \frac{\Phi(s)^{s^2}}{\varphi(0)^s}, \quad \alpha \in (0, 1],$$

$$\mathcal{L}_m\left\{{}^*D^{2\alpha}\varphi(t)\right\} = \frac{\Phi(s)^{s^2}}{\varphi(0)^s.{}^*D^\alpha\varphi(0)}, \quad \alpha \in (1, 2],$$

Definition 3.10. If $\mathcal{L}_m\{\varphi\} = \Phi$, then $\mathcal{L}_m^{-1}\{\Phi\}$ is referred to as the inverse multiplicative Laplace transform of φ.

4. Illustrative Examples

In this section, we discuss the significance and effectiveness of the proposed method by presenting several examples.

Example 4.1. Consider the following multiplicative conformable fractional Cauchy problem with the initial condition

$$\begin{cases} {}^*D^{\alpha}\varphi(t) = \varphi(t)^{-c}, \quad t > 0, \quad \alpha \in (0,1], \\ \varphi(0) = e^{-b}. \end{cases}$$

Taking the multiplicative Laplace transform of the multiplicative conformable fractional problem and applying the initial condition, we obtain

$$\begin{aligned} \mathcal{L}_m\{{}^*D^{\alpha}\varphi(t)\} = \mathcal{L}_m\{\varphi(t)^{-c}\} \quad &\Rightarrow \quad \frac{\Phi(s)^s}{e^{-b}} = \Phi(s)^{-c} \\ &\Rightarrow \quad \Phi(s)^{s+c} = e^{-b} \\ &\Rightarrow \quad \Phi(s) = e^{\frac{-b}{s+c}}. \end{aligned}$$

By applying the inverse transform, we obtain

$$\begin{aligned} \mathcal{L}_m^{-1}\{\Phi(s)\} &= \mathcal{L}_m^{-1}\left\{e^{-b\frac{1}{s+c}}\right\}, \\ \varphi(t) &= \exp\left(-be^{-c\frac{t^{\alpha}}{\alpha}}\right). \end{aligned}$$

This is known as the Gompertz function, which describes a demographic model.

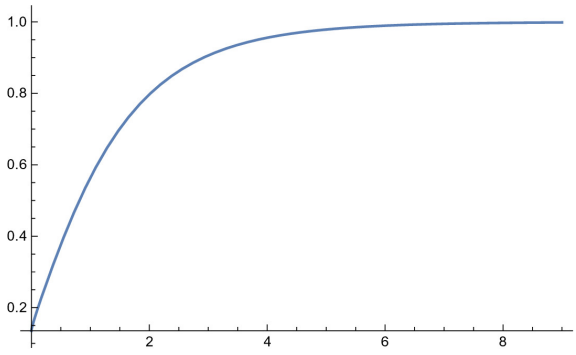


Figure 1. Plot for example (4.1) for $c = 1, b = 2$ and $\alpha = 0.9$.

Example 4.2. Consider the the following multiplicative conformable fractional differentia equation with the initial condition

$$\begin{cases} {}^{*}D^{2\alpha}\varphi(t).\varphi^{-1}(t) = \exp(\cos\left(\frac{t^{\alpha}}{\alpha}\right)), \quad t > 0, \quad \alpha \in (0,1], \\ \varphi(0) = 1. \end{cases} \tag{4.1}$$

Using the multiplicative Laplace transform the conformable fractional problem and using multiplicative Laplace transforms of elementary functions, we have

$$\begin{aligned} \mathcal{L}_m\left\{{}^{*}D^{2\alpha}\varphi(t)\right\}\mathcal{L}_m\{\varphi(t)\}^{-1} &= \mathcal{L}_m\left\{\cos\left(\frac{t^{\alpha}}{\alpha}\right)\right\} \\ \Rightarrow \quad \frac{\Phi(s)^{s^2}}{\varphi(0)}\Phi(s)^{-1} &= \exp\left(\frac{s}{s^2+1}\right) \\ \Rightarrow \quad \Phi(s)^{s^2-1} &= \exp\left(\frac{s}{s^2+1}\right) \\ \Rightarrow \quad \Phi(s) &= \exp\left(\frac{s}{(s^2+1)(s^2-1)}\right). \end{aligned}$$

By applying the inverse transform and using multiplicative Laplace transforms of function obtained, we have

$$\begin{aligned} \mathcal{L}_m^{-1}\{\Phi(s)\} &= \mathcal{L}_m^{-1}\left\{\exp\left(\frac{s}{(s^2+1)(s^2-1)}\right)\right\}, \\ \varphi(t) &= \exp\left[\frac{1}{2}\cosh\left(\frac{t^{\alpha}}{\alpha}\right) - \frac{1}{2}\cos\left(\frac{t^{\alpha}}{\alpha}\right)\right]. \end{aligned}$$

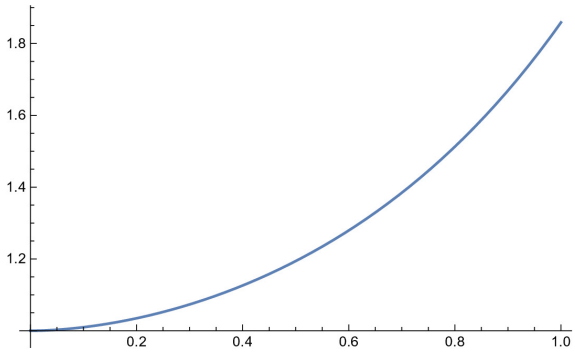


FIGURE 2. Plot for example (4.2) for $\alpha = 0.9$.

Example 4.3. Consider the the following multiplicative conformable fractional differentia equation with the initial condition

$$\begin{cases} {}^*D^{2\alpha}\varphi(t).\varphi^{-4}(t) = 1, \quad t > 0, \quad \alpha \in [1,2), \\ \varphi(0) = e, {}^*D^{\alpha}\varphi(0) = 1. \end{cases}$$

Taking the multiplicative Laplace transform of both sides of problem and using multiplicative Laplace transforms of elementary functions, we have

$$\begin{aligned} & \mathcal{L}_m \left\{ {}^*D^{2\alpha}\varphi(t) \right\} \mathcal{L}_m\{\varphi(t)\}^{-4} = \mathcal{L}_m\{1\} \\ \Rightarrow \quad & \frac{\Phi(s)^{s^2}}{\varphi(0)^{s*}D^{\alpha}\varphi(0)} \Phi(s)^{-4} = 1 \\ \Rightarrow \quad & \Phi(s)^{s^2-4} = \exp(s) \\ \Rightarrow \quad & \Phi(s) = \exp\left(\frac{s}{s^2-4} \right). \end{aligned}$$

By applying the inverse multiplicative Laplace transform, we have

$$\mathcal{L}_m^{-1}\{\Phi(s)\} = \mathcal{L}_m^{-1}\left\{ \exp\left(\frac{s}{s^2-4} \right) \right\},$$

therefore

$$\varphi(t) = \exp\left[\cosh\left(\frac{2t^{\alpha}}{\alpha} \right) \right].$$

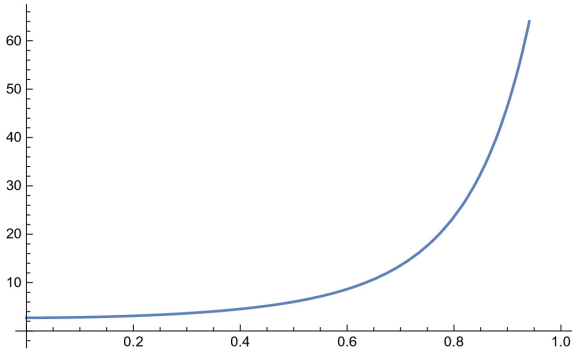


FIGURE 3. Plot for example (4.3) for $\alpha = 0.9$.

5. Conclusion

In this paper, we first introduced the fundamental definitions of multiplicative differential equations and conformable fractional differential equations, highlighted the key features utilized throughout the discussion. We defined multiplicative fractional differential equations with conformable derivatives. We also extended the Laplace transform of the conformable fractional derivative to include the conformable multiplicative fractional derivative. Finally, by presenting several significant and applicable examples, we derived their analytical solutions using the established Laplace transform and illustrated these solutions graphically. This study demonstrated the significance and effectiveness of the proposed method, which can be utilized to solve multiplicative fractional differential Cauchy problems with conformable derivatives.

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[1] Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran.
Email address: m.jahanshahi@azaruniv.ac.ir

[2] Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran.
Email address: reza.daei@azaruniv.ac.ir