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Basic structures of Riemannian Complex Fuzzy Manifolds

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Abstract. In this paper, we fuzzify the basic concepts of the ordinary n-manifold, using the concept of Riemannian complex fuzzy sets (RCFS). We first recall the concept of a RCFS which provides a mathematical framework for describing membership in a set in terms of a complex-valued function and a real-valued function $\left(\hbar_{\wp}(z)e^{i\circledS_{\wp}(z)}, \ell_{\wp}(z)\right)$. Then we define the image and preimage of a Riemannian complex fuzzy set under a mapping and define the Riemannian complex fuzzy topological space $(Z, \Im)$ with Riemannian complex gradation of openness, where Z is itself a Riemannian complex fuzzy subset of a nonempty set M $(RCGfts)$. As a usefull example we introduce Riemannian complex fuzzy Euclidean topology of dimension n. Also we define RCG-related functions, $RCGP$-related functions, RCG-continuous functions, and RCG-homeomorphisms between two $RCGftss$. Then we introduce C^{∞} Riemannian complex fuzzy n-manifolds $(Z, \Im)$ with Riemannian complex gradation of openness such as RCG-fuzzy quotient manifolds and C^{∞} RCG-fuzzy mappings of them such as RCG-fuzzy immersion and RCG-fuzzy imbedding functions. Finally, we define imbedded RCG-fuzzy submanifolds as well as regular RCG-fuzzy submanifolds and deduce several theorems concerning the relations between them.

1. Introduction

Fuzzy theories have been proposed to deal with uncertainty and imprecision, and are are applicable to almost all real world problems. For example, fuzzy set theory (FS) by Zadeh [57], rough set theory by

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Pawlak [36], theory of intuitionistic fuzzy sets by Attanassov [5], theory of Pythagorean fuzzy sets by Peng [37], theory of complex fuzzy sets by Ramot et al. [41], theory of soft sets by Molodtsov [33], and the theory of fuzzy soft sets by Maji et al. [30]. These models are used in many cases in situations of uncertainty such as engineering, computer science, decision-making problems, networking, pattern recognition, and many other scientific fields. Some applications of fuzzy theories can be seen in the following articles:

Alimoradpour et al. [3] discussed different approaches using genetic algorithm for optimal fuzzy rule-based system. Raj et al. [51] designed a neural network learning machine (ELM)-based fuzzy controller and used it to improve speed regulation of permanent magnet synchronous motor. Meng et al. [32] proposed a novel type 2 interval fuzzy multi-attribute group decision-making method, to evaluate sponge city construction applications. Ibrahim et al. [16] presented the theory of (3, 2)-fuzzy sets and propounded their applications in topology and optimal choices. Gehrke et al. [11] proposed the theory of interval-valued fuzzy sets, Qiu et al. [39] discussed polynomial regression interval-valued fuzzy systems and Yiarayong [56] introduced an interval-valued fuzzy soft semigroups that combines the concepts of soft interval-valued fuzzy sets on semigroups. Atanasov [4] introduced the concept of intuitionistic fuzzy sets (IFS), which is a very useful generalization of fuzzy sets. As an extension of intuitionistic fuzzy set, Pythagorean fuzzy set (PFS) was presented by Yager [54]. Xian et al. introduced an extension of the Pythagorean fuzzy time series model under the PFS environment, and their applications in predicting new COVID-19 cases were discussed in [53].

Many physical quantities are complex-valued: complex amplitude and impedance in electrical engineering, wave function in quantum mechanics, etc. Complex fuzzy sets are a unique framework compared to traditional fuzzy sets. The support of complex fuzzy set is unlimited and can include any type of object such as a number, name etc., which of course is a complex number. Ramot [41] introduced the concept of complex fuzzy sets (CFS), where the membership function instead of being a real valued function with the range of $[0, 1]$, is replaced by a complex-valued function of the form $\mu_{_A}(z) = r_{_A}(z)e^{i\omega_{_A}(z)}$, where $r_{_A}$ and $\omega_{_A}$ are both real valued functions and $\mu_{_A}$ has a value in the range of complex unit circle. As explained in [41], one of the important properties of complex fuzzy sets is the existence of phase and membership. This leads to the wave-like properties of complex fuzzy sets, which can produce constructive and destructive interference depending on the phase value. Later in the works of other mathematicians, various complex fuzzy structures, were defined

and their application were discussed, such that [1, 2, 9, 13, 14, 17–22, 24, 28, 29, 40, 42, 47, 48, 50, 59]. In [35] we defined (anti) complex fuzzy subspace of a k-vector space, under T-norm (C-conorm) and investigated various operations between T-complex fuzzy subspaces. Since, at least in some cases, it is necessary to add a third dimension to the expression of membership term, the concept of Riemannian complex fuzzy sets (RCFSs) was introduced in [58]. The RCFS, $\wp$ is characterized by a membership function, $\mu_{\wp}(z)$, which assigns a grade of membership of the form $\left(\hbar_{\wp}(z)e^{i\circledS_{\wp}(z)}, \ell_{\wp}(z)\right)$ $(i = \sqrt{-1})$ to each element z in the universe of discourse. The RCFS is the combination of a CFS and a FS. The value $\mu_{\wp}(z)$ is denoted by the three variables, $\hbar_{\wp}(z)$, $\circledS_{\wp}(z)$, and $\ell_{\wp}(z)$ with $\hbar_{\wp}(z)$, $\ell_{\wp}(z) \in [0; 1]$, and $\circledS_{\wp}(z) \in [0, 2\pi]$. In addition, Zeeshan et al. [58] developed a new algorithm using RCFSs for applications in signals and systems, in which they identify a reference signal from large input signals detected by a system.

In this paper, we define the image and inverse image of an RCFS under a function and show they are also Riemannian complex fuzzy sets. We also present some theorems and numerical examples. Since Chang [6] introduced the concept of fuzzy topological spaces, many authors like Wong [52], E. Lowen et al. [27], R. Lowen [25, 26], Shostak [44–46], Chattopadhyay et al. [7, 8], Gregori [12], Stadler et al. [49], and Ibedou [15] presented various kinds of fuzzy topological spaces.

While almost all research on fuzzy topological spaces and fuzzy manifolds has focused on a crispt set, in this paper, we show the possibility of improving the current definitions using our previous work [34]. We define the Riemannian complex fuzzy topological space $(Z, \Im)$ with a Riemannian complex gradation of openness, where Z is itself a Riemannian complex fuzzy subset of a nonempty set N, (briefly $RCGfts$). For example, we present a Riemannian complex fuzzy Euclidean topological space with Riemannian complex gradation of openness $(\tilde{1}_{\mathbb{R}^n}, \Im_{RCn})$. We also define RCG-related functions, $RCGP$-related functions, RCG-continuous functions, and RCG-homeomorphisms between two $RCGftss$. In the following, we introduce C^{∞} Riemannian complex fuzzy manifolds $(Z, \Im)$ with the Riemannian complex gradation of openness, with a different interpretation of [10] and obtain C^{∞} n-premanifolds from them. Note that these manifolds are completely different from the Riemannian manifolds presented in books such as [31] and [23] or in articles such as [38] and [43]. We define RCG-fuzzy quotient manifolds and then study the C^{∞} RCG-fuzzy mappings of RCG-fuzzy manifolds. Then we prove the RCG-fuzzy rank theorem. We discuss RCG-fuzzy immersion functions and LG-fuzzy imbedding functions. In the following, we define RCG-fuzzy immersions (RCG-fuzzy submersions) and RCG-fuzzy

submanifolds of RCG-fuzzy manifolds. We point out the difference between ordinary topological manifolds and Riemannian complex fuzzy topological manifolds:

Let $g \in RCGRF(Z,Y)$ be a RCG-fuzzy immersion. Then for each $p \in Z$, there is not necessarily a RCG-neighborhood $\wp$ of p, such that $g|_{supp\wp}$ is a RCG-fuzzy imbedding. Finally, we define imbedded fuzzy-RCG submanifolds as well as regular fuzzy-RCG submanifolds, and then some theorems about the relations between them are derived.

2. Preliminaries

Definition 2.1. [41] Let N be a nonempty set. A complex fuzzy set (CFS) B on N is an object having the form $B = \{(z, \mu_{_B}(z))|z \in N\}$, where $\mu_{_B}$ denotes the degree of membership function that assigns each element $z \in N$, a complex number $\mu_{_B}(z)$ lies within the unit circle in the complex plane. We shall assume that $\mu_{_B}(z)$ will be represented by $r_{_B}(z)e^{i\omega_{_B}(z)}$, where $i = \sqrt{-1}$, and $r_{_B} : N \to [0,1]$ and $\omega_{_B} : N \to [0, 2\pi]$. The term $r_{_B}(z)$ is said to be phase term and $\omega_{_B}(z)$ is said to be amplitude term.

Definition 2.2. Let $B = (z, \mu_{_B}(z))$ and $D = (z, \mu_{_D}(z))$ be complex fuzzy subsets of a nonempty set N. We define complex fuzzy subsets $B \cap D,\ B \cup D$ by

$$\begin{array}{ll} \mu_{_{B\cap D}}(z) = r_{_{B\cap D}}(z)e^{i\omega_{_{B\cap D}}(z)}, & r_{_{B\cap D}}(z) = T\left(r_{_B}(z), r_{_D}(z)\right), \\ \omega_{_{B\cap D}}(z) = \min\left(\omega_{_B}(z), \omega_{_D}(z)\right), & \mu_{_{B\cup D}}(z) = r_{_{B\cup D}}(z)e^{i\omega_{_{B\cup D}}(z)}, \\ r_{_{B\cup D}}(z) = B\left(r_{_B}(z), r_{_D}(z)\right), & \omega_{_{B\cup D}}(z) = \left(\omega_{_B}(z), \omega_{_D}(z)\right). \end{array}$$

Complex fuzzy subsets $B + D$ and $\gamma.B$ of a vector space V, for each $\gamma \in k,\ z \in Z$, are defined by

$$\mu_{_{B+D}}(z) = r_{_{B+D}}(z)e^{i\omega_{_{B+D}}(z)},$$

$$r_{_{B+D}}(z) = \begin{cases} \sup_{z=b+d}\left\{T\left(r_{_B}(b), r_{_D}(d)\right)\right\} & \text{if } z = b+d \\ 0 & \text{elsewhere} \end{cases}$$

$$\omega_{_{B+D}}(z) = \begin{cases} \sup_{z=b+d}\left\{\min\left(\omega_{_B}(b), \omega_{_D}(d)\right)\right\} & \text{if } z = b+d, \\ 0 & \text{elsewhere} \end{cases}$$

and $\mu_{\gamma.B}(z) = r_{\gamma.B}(z)e^{i\omega_{\gamma.B}(z)}$,

$$r_{_{\gamma.B}}(z) = \begin{cases} r_{_B}(\frac{1}{\gamma}z) & \text{if } \gamma \neq 0 \\ 1 & \text{if } \gamma = 0,\ z = 0 \\ 0 & \text{if } \gamma = 0,\ z \neq 0 \end{cases}$$

$$\omega_{_{\gamma.B}}(z) = \begin{cases} \omega_{_B}\left(\frac{1}{\gamma}z\right) & \text{if } \gamma \neq 0 \\ 0 & \text{if } \gamma = 0. \end{cases}$$

Further if $B \cap D = \tilde{0}$, then $B + D$ is said to be the direct sum and denoted by $B \oplus D$.

3. Riemannian Complex Fuzzy Sets

In this section, we give the novel concept of a RCFS which is the more generalized form of a CFS. The RCFS provides a mathematical framework for describing membership in a set in terms of a complex-valued function and a real-valued function.

Definition 3.1 ([58])**.** A Riemannian complex fuzzy set (RCFS) $\wp$, defined on an universe of discourse N, is characterized by a membership function $\mu_{\wp}(z)$ for any element $z \in N$. The membership function of a Riemannian complex fuzzy set is defined as:

$$\mu_{\wp}(z) = \left(\hbar_{\wp}(z)e^{i\circledS_{\wp}(z)}, \ell_{\wp}(z)\right)$$

where $\hbar_{\wp}(z)$ and $\ell_{\wp}(z) \in [0,1]$, $i = \sqrt{-1}$, and $\circledS_{\wp}(z) \in [0, 2\pi]$.

The RCFS may be represented as:

$$\wp = \{(z, \alpha_{\wp}(z), \ell_{\wp}(z)) \mid z \in N\}$$

where $\alpha_{\wp}(z) = \hbar_{\wp}(z)e^{i\circledS_{\wp}(z)}$. So $\alpha_{\wp}$ represents a complex fuzzy set and $\ell_{\wp}$ represents a fuzzy set.

Definition 3.2 ([58])**.** Let $\wp_1$ and $\wp_2$ be two RCFSs and let

$$\mu_{\wp_1}(z) = \left(\hbar_{\wp_1}(z)e^{i\circledS_{\wp_1}(z)}, \ell_{\wp_1}(z)\right), \ \mu_{\wp_2}(z) = \left(\hbar_{\wp_2}(z)e^{i\circledS_{\wp_3}(z)}, \ell_{\wp_2}(z)\right)$$

their MFs respectively. Now $\wp_1$ is said to be Riemannian complex fuzzy subset of $\wp_2$ and write $\wp_1 \leq \wp_2$ or $\wp_1 \subset \wp_2$ if

i) $\hbar_{\wp_1}(z) \leq \hbar_{\wp_2}(z)$, $\circledS_{\wp_1}(z) \leq \circledS_{\wp_2}(z)$;
ii) $\ell_{\wp_1}(z) \leq \ell_{\wp_2}(z)$.

for all $z \in M$.

Definition 3.3 ([58])**.** Let $\wp$ be a RCFSs over N. Then, the complement of $\wp$, denoted by $\wp^c$ and is defined as:

$$\wp^c = \left\{\left(z, \alpha^c_{\wp}(z), \ell^c_{\wp}(z)\right) \mid z \in N\right\},$$

where $\alpha^c_{\wp}$ and $\ell^c_{\wp}$ denote the complement of traditional CFSs and traditional FSs, respectively. So

$$\mu_{\wp^c} = \left((1 - \hbar_{\wp_1}(z))\ e^{i\circledS_{\wp}(z)}, (1 - \ell_{\wp_1}(z)\right).$$

Definition 3.4 ([58])**.** Let $\wp_1$ and $\wp_2$ be two RCFSs, and let $\mu_{\wp_1}(z), \mu_{\wp_2}(z)$ denote the MFs of $\wp_1$ and $\wp_2$ respectively. Then, Riemannian complex fuzzy union, intesection, and simple difference of these two RCFSs are respectively, defined by:

$$\begin{aligned}
\mu_{\wp_1\cup\wp_2}(z) &= \mu_{\wp_1}(z) \cup \mu_{\wp_2}(z) \\
&= \Big(\max\{\hbar_{\wp_1}(z), \hbar_{\wp_2}(z)\}\ e^{i\max\{\mathbb{S}_{\wp_1}(z), \mathbb{S}_{\wp_2}(z)\}}, \\
&\qquad \max\{\ell_{\wp_1}(z), \ell_{\wp_2}(z)\}\Big), \\
\mu_{\wp_1\cap\wp_2}(z) &= \mu_{\wp_1}(z) \cap \mu_{\wp_2}(z) \\
&= \Big(\min\{\hbar_{\wp_1}(z), \hbar_{\wp_2}(z)\}\ e^{i\min\{\mathbb{S}_{\wp_1}(z), \mathbb{S}_{\wp_2}(z)\}}, \\
&\quad \min\{\ell_{\wp_1}(z), \ell_{\wp_2}(z)\}\Big), \\
\mu_{\wp_1\setminus\wp_2}(z) &= \mu_{\wp_1}(z) \cap \mu_{\wp_2^c}(z) \\
&= \Big(\min\{\hbar_{\wp_1}(z), \hbar_{\wp_2^c}(z)\}\ e^{i\min\{\mathbb{S}_{\wp_1}(z), \mathbb{S}_{\wp_2^c}(z)\}}, \\
&\quad \min\{\ell_{\wp_1}(z), \ell_{\wp_2^c}(z)\}\Big) \\
&= \Big(\min\{\hbar_{\wp_1}(z), 1-\hbar_{\wp_2}(z)\}\ e^{i\min\{\mathbb{S}_{\wp_1}(z), \mathbb{S}_{\wp_2}(z)\}}, \\
&\quad \min\{\ell_{\wp_1}(z), 1-\ell_{\wp_2}(z)\}\Big).
\end{aligned}$$

Example 3.5. Let $N = \{z_1, z_1, z_3\}$ and $\wp_1,\ \wp_2$ be two RCFSs on it,

$$\begin{aligned}
\wp_1 &= \{(z_1, 0.7e^{i0.7\pi}, 0.9),\ (z_2, 0.4e^{i2\pi}, 0.8),\ (z_3, 0.9e^{i0.8\pi}, 0.7)\} \\
\wp_2 &= \{(z_1, 0.4e^{i0.9\pi}, 0.5),\ (z_2, 0.6e^{i1.4\pi}, 0.9),\ (z_3, 0.2e^{i0.3\pi}, 0.8)\}
\end{aligned}$$

Then their operations are defined as follows:

$$\begin{aligned}
\wp_1 \cup \wp_2 &= \{(z_1, 0.7e^{i0.9\pi}, 0.9),\ (z_2, 0.6e^{i2\pi}, 0.9),\ (z_3, 0.9e^{i0.8\pi}, 0.8)\} \\
\wp_1 \cap \wp_2 &= \{(z_1, 0.4e^{i0.7\pi}, 0.5),\ (z_2, 0.4e^{i1.4\pi}, 0.8),\ (z_3, 0.2e^{i0.3\pi}, 0.7)\} \\
\wp_1 \setminus \wp_2 &= \{(z_1, 0.3e^{i0.7\pi}, 0.4),\ (z_2, 0e^{i1.4\pi}, 0),\ (z_3, 0.7e^{i0.3\pi}, 0)\} \\
\wp_1^c &= \{(z_1, 0.3e^{i0.7\pi}, 0.1),\ (z_2, 0.6e^{i2\pi}, 0.2),\ (z_3, 0.1e^{i0.8\pi}, 0.3)\}
\end{aligned}$$

Definition 3.6. Let $g : N \to N'$ be a map and $\wp$ and $\wp'$ be two RCFSs on N and N' respectively. Then the inverse image of $\wp'$ under g, is a Riemannian complex fuzzy subset, denoted by $g^{-1}[\wp']$ on N, defined by

$$\begin{aligned}
\mu_{g^{-1}[\wp']}(z) &= \Big(\hbar_{g^{-1}[\wp']}(z)\ e^{i\mathbb{S}_{g^{-1}[\wp']}(z)}, \ell_{g^{-1}[\wp']}(z)\Big) \\
&= \Big(\hbar_{\wp'}(g(z)\ e^{i\mathbb{S}_{\wp'}(g(z))}, \ell_{\wp'}(g(z))\Big)
\end{aligned}$$

for all $z \in N$ and the image of $\wp$ under g is a Riemannian complex fuzzy set, denoted by $g[\wp]$ on N', defined by

$$\mu_{g[\wp]}(y) = (\hbar_{g[\wp]}(y)e^{i\circledS_{f[\wp]}(y)}, \ell_{g[\wp]}(y)),$$

where

$$\hbar_{g[\wp]}(y) = \begin{cases} \sup\{\hbar_{\wp}(z)|\ z \in g^{-1}(y)\} & \text{if } w \in g(N) \\ 0 & \text{if } w \notin g(N) \end{cases}$$

$$\circledS_{g[\wp]}(y) = \begin{cases} \sup\{\circledS_{\wp}(z)|\ z \in g^{-1}(y)\} & \text{if } y \in g(N) \\ 0 & \text{if } y \notin g(N) \end{cases}$$

$$\ell_{g[\wp]}(y) = \begin{cases} \sup\{\ell_{\wp}(z)|\ z \in g^{-1}(y)\} & \text{if } w \in g(N) \\ 0 & \text{if } w \notin g(N) \end{cases}$$

for all $y \in N'$

Example 3.7. Let $N = \{z_1, z_2, z_3\}$, $N' = \{y_1, y_2, y_3\}$ and $g : N \to N'$ be a map defined by $g(z_1) = y_3$, $g(z_2) = f(z_3) = y_1$. Let $\wp$ be a RCFSs on N:

$$\wp = \{(z_1, 0.7e^{i0.7\pi}, 0.9),\ (z_2, 0.4e^{i2\pi}, 0.8),\ (z_3, 0.9e^{i0.8\pi}, 0.7)\}$$

Then $g[\wp]$ on N' is defined as follows:

$$g[\wp] = \{(w_1, 0.9e^{i2\pi}, 0.8),\ w_2, 0, 0),\ (w_3, 0.7e^{i0.7\pi}, 0.9)\}$$

Let $\wp'$ be a RCFSs on N':

$$\wp' = \{(w_1, 0.2e^{i\pi}, 0.9),\ (w_2, 0.9e^{i0.3\pi}, 0.7),\ (w_3, 0.1e^{i2\pi}, 0.5)\}$$

Then $g^{-1}[\wp']$ on N is defined as follows:

$$g^{-1}[\wp'] = \{(z_1, 0.1e^{i2\pi}, 0.5),\ (z_2, 0.2e^{i\pi}, 0.9),\ (z_3, 0.2e^{i\pi}, 0.9)\}$$

Theorem 3.8. *Let $g : N \to N'$ be a map and $\wp$ and $\wp'$ be two RCFSs on N and N' respectively. Then*

1) $g^{-1}[\wp'^c] = (g^{-1}[\wp'])^c$.
2) $(g[\wp])^c \subset g[\wp^c]$.
3) *Ig $\wp'_1 \subset \wp'_2$, then $g^{-1}[\wp'_1] \subset g^{-1}[\wp'_2]$ where $\wp'_1$ and $\wp'_2$ are two RCFSs on N'.*
4) *Ig $\wp_1 \subset \wp_2$, then $g[\wp_1] \subset g[\wp_2]$ where $\wp_1$ and $\wp_2$ are two RCFSs on N.*
5) $g[g^{-1}[\wp']] \subset \wp'$.
6) $\wp \subset g^{-1}[g[\wp]]$.

Proof. (1) For each $z \in N$ we have

$$\hbar_{g^{-1}[\wp'^c]}(z) = \hbar_{\wp'^c}(g(z)) = 1 - \hbar_{\wp'}(g(z)) = 1 - \hbar_{g^{-1}[\wp']}(z) = \hbar_{(g^{-1}[\wp'])^c}(z).$$

Similarly, we can show $\ell_{g^{-1}[\wp'^c]}(z) = \ell_{(g^{-1}[\wp'])^c}(z)$. Also we see

$$Ⓢ_{g^{-1}[\wp'^c]}(z) = Ⓢ_{\wp'^c}(g(z)) = Ⓢ_{\wp'}(g(z)) = Ⓢ_{g^{-1}[\wp']}(z) = Ⓢ_{(g^{-1}[\wp'])^c}(z)$$

Therefore, $g^{-1}[\wp'^c] = (g^{-1}[\wp'])^c$.
(2) For each $w \in N'$ that $g^{-1}(w) \neq \phi$, we have

$$\begin{aligned}\hbar_{g[\wp^c]}(z) &= \sup\{\hbar_{\wp^c}(z)|\ z \in g^{-1}(w)\}\\ &= \sup\{1 - \hbar_{\wp}(z)|\ z \in g^{-1}(w)\}\\ &\geq 1 - \sup\{\hbar_{\wp}(z)|\ z \in g^{-1}(w)\}\\ &= 1 - \hbar_{g[\wp]}(z) = \hbar_{(g[\wp])^c}(z)\end{aligned}$$

If $g^{-1}(w) = \phi$, the proof is clear. Using a similar method, we can show $Ⓢ_{g[\wp^c]}(z) = Ⓢ_{(g[\wp])^c}(z)$ and $\ell_{g[\wp^c]}(z) = \ell_{(g[\wp])^c}(z)$. Hence $(g[\wp])^c \subset g[\wp^c]$.
(3) Assume that $\wp'_1 \subset \wp'_2$, then for any $z \in N$, we have $\hbar_{g^{-1}[\wp'_1]} = \hbar_{\wp'_1}(g(z)) \leq \hbar_{\wp'_2}(g(z)) = \hbar_{g^{-1}[\wp'_2]}$.
Similarly we obtain $Ⓢ_{g^{-1}[\wp'_1]} \leq Ⓢ_{g^{-1}[\wp'_2]}$ and $\ell_{g^{-1}[\wp'_1]} \leq \ell_{g^{-1}[\wp'_2]}$. Therefore $g^{-1}[\wp'_1] \subset g^{-1}[\wp'_2]$.
(4) Suppose $\wp_1 \subset \wp_2$ and $w \in N'$. If $g^{-1}(w) = \phi$, the proof is easy. So, suppose that $g^{-1}(y) \neq \phi$. Then

$$\hbar_{g[\wp_1]}(z) = \sup_{z \in g^{-1}(w)} \hbar_{\wp_1}(z) \leq \sup_{z \in g^{-1}(w)} \hbar_{\wp_2}(z) = \hbar_{g[\wp_2]}(z)$$

In a similar way, we have $Ⓢ_{g[\wp_1]}(z) \leq Ⓢ_{g[\wp_2]}(z)$ and $\ell_{g[\wp_1]}(z) \leq \ell_{g[\wp_2]}(z)$. Thus $g[\wp_1] \subset g[\wp_2]$
(5) For any $w \in N'$ that $g^{-1}(w) \neq \phi$, we see

$$\begin{aligned}\hbar_{g[g^{-1}[\wp']]}(w) &= \sup_{z \in g^{-1}(w)} \hbar_{g^{-1}[\wp']}(z)\\ &= \sup_{z \in g^{-1}(w)} \hbar_{\wp'}(g(z))\\ &= \sup_{z \in g^{-1}(w)} \hbar_{\wp'}(y)\\ &= \hbar_{\wp'}(w)\end{aligned}$$

Similarly , we have $Ⓢ_{g[g^{-1}[\wp']]}(w) = Ⓢ_{\wp'}(w)$ and $\ell_{g[g^{-1}[\wp']]}(w) = \ell_{\wp'}(w)$.
If $g^{-1}(w) = \phi$, we have $\hbar_{g[g^{-1}[\wp']]}(w) = 0 \leq \hbar_{\wp'}(w)$.
Using the same maner, we have

$$Ⓢ_{g[g^{-1}[\wp']]}(w) = 0 \leq Ⓢ_{\wp'}(w), \qquad \ell_{g[g^{-1}[\wp']]}(w) = 0 \leq \ell_{\wp'}(w).$$

Therefore $g[g^{-1}[\wp']] \subset \wp'$.
(6) For each $z \in N$ we have

$$\hbar_{g^{-1}[g[\wp]]}(z) = \hbar_{g[\wp]}(g(z)) = \sup_{z \in g^{-1}(g(z))} \hbar_{\wp}(z) \geq \hbar_{\wp}(z)$$

Similarly, we have $Ⓢ_{g^{-1}[g[\wp]]}(z) \geq Ⓢ_{\wp}(z)$ and $\ell_{g^{-1}[g[\wp]]}(z) \geq \ell_{\wp}(z)$. Hence $\wp \subset g^{-1}[g[\wp]]$. □

Theorem 3.9. *Let $g : N \to N'$ be a map. Then, The following axioms hold.*

1) *For any family $\{\wp_j\}_{j \in J}$ of RCFSs on N, we have*

$$g\left[\bigcup_{j \in J} \wp_j\right] = \bigcup_{j \in J} g[\wp_j]$$

1) *For any family $\{\wp'_j\}_{j \in J}$ of RCFSs on N', we have*

$$g^{-1}\left[\bigcup_{j \in J} \wp'_j\right] = \bigcup_{j \in J} g^{-1}[\wp'_j]$$

3) *For any two RCFSs, $\wp_1$ and $\wp_2$ on N, we have*

$$g[\wp_1 \cap \wp_2] = g[\wp_1] \cap g[\wp_2]$$

4) *For any family $\{\wp'_j\}_{j \in J}$ of RCFSs on N', we have*

$$g^{-1}\left[\bigcap_{j \in J} \wp'_j\right] = \bigcap_{j \in J} g^{-1}[\wp'_j]$$

4. Topology with respect to Riemannian Complex Fuzzy Sets

Definition 4.1. Let Z be a RCFS of N and

$$\mu_Z(z) = (\hbar_Z(z)e^{i Ⓢ_Z(z)}, \ell_Z(z))$$

its MF. By $(S^1 \times I)_Z^N$, we denote the set of all RCFSs of N, which are less or equal to Z (called Riemannian complex fuzzy subsets of Z). Let τ be a family of Riemannian complex fuzzy subsets of Z, satisfying three conditions:

1) $Z,\ \phi \in \tau$,
2) $\wp_1,\ \wp_2 \in \tau \ \Rightarrow\ \wp_1 \cap \wp_2 \in \tau$,
3) $\{\wp_j\}_{j \in J} \subseteq \tau \ \Rightarrow\ \bigcup_{j \in J} \wp_j \in \tau$,

then (Z,τ) is called a Riemannian complex fuzzy topological space (RC-fts).
Let $\wp \in (S^1 \times I)^N_Z$. If $\wp \in \tau$, then $\wp$ is called an open RCFS, and if $(Z\backslash\wp) \in \tau$, it is called a closed RCFS.

Example 4.2. Let $N = \mathbb{R}^n$ and $Z = \tilde{1}$, $\mu_{\tilde{1}}(z) = (1e^{i2\pi}, 1)$. Let $B(a,r,b,c,d)$ be a RCFS of N with

$$\mu_{B(a,r,b,c,d)}(z) = \begin{cases} \big(b(z)e^{ic(z)}, d(z)\big) & \text{if } z \in B(a,r), \\ \tilde{0}(z) = (0e^{i0}, 0) & \text{otherwise,} \end{cases}$$

which $b, d : B(a,r) \to [0,1]$, and $c : B(a,r) \to [0,2\pi]$ are three arbitrary functions. The fuzzy topology induced by

$$\beta_{RCn} = \{B(a,r,b,c,d),\ a \in \mathbb{R}^n,\ r \in \mathbb{R}^+,$$
$$b, d : B(a,r) \to [0,1], c : B(a,r) \to [0,2\pi], \text{are functions}\}$$

is called Riemannian complex fuzzy Euclidean topology of dimension n and is denoted by τ_{RCn}.

Definition 4.3. Let (Z,τ) be a Riemannian complex fuzzy topological space and $\wp \in (S^1 \times I)^N_Z$. Then RC-fuzzy interior and RC-fuzzy closure of $\wp$ are respectively, defined by

(1) $int(\wp) = \bigcup\{\wp_0 : \wp_0 \in \tau,\ \wp_0 \subset \wp\}$,
(2) $cl(\wp) = \bigcap\{\wp_1 : (Z\backslash\wp_1) \in \tau,\ \wp \subset \wp_1\}$.

Theorem 4.4. *Let (Z,τ) be a RC-fts and $\wp_1, \wp_2 \in (S^1 \times I)^N_Z$. Then we have:*

(1) $int(\wp_1) \subset \wp_1$ *and* $\wp_1 \subset cl(\wp_1)$.
(2) *If* $\wp_1 \subset \wp_2$, *then* $int(\wp_1) \subset int(\wp_2)$ *and* $cl(\wp_1) \subset cl(\wp_2)$
(3) $\wp_1$ *is an open RCFS if and only if* $\wp_1 = int(\wp_1)$.
(4) $\wp_1$ *is a closed RCFS if and only if* $\wp_1 = cl(\wp_1)$.

Proof. (1) and (2) are clear. Using by Definition 4.3, obviously we have (3) and (4). □

Definition 4.5. Let $\mathfrak{I} : (S^1 \times I)^N_Z \to S^1 \times I$ be a mapping represented by

$$\mathfrak{I}(\wp) = \left(\hbar_{\jmath}(\wp)e^{i\circledS_{\jmath}(\wp)}, \ell_{\jmath}(\wp)\right), \quad \forall \wp \in RCI^N_Z$$

where $\hbar_{\jmath}, \ell_{\jmath} : (S^1 \times I)^N_Z \to [0,1]$ and $\circledS_{\jmath} : (S^1 \times I)^N_Z \to [0,2\pi]$ satisfy:

(i) $\mathfrak{I}(Z) = \mathfrak{I}(\tilde{0}) = (1e^{i2\pi}, 1)$,
(ii) $\forall \wp_1,\ \wp_2 \in (S^1 \times I)^N_Z$

$$\hbar_{\jmath}(\wp_1 \cap \wp_1) \geq \min\left(\hbar_{\jmath}(\wp_1), \hbar_{\jmath}(\wp_1)\right),$$
$$\circledS_{\jmath}(\wp_1 \cap \wp_1) \geq \min\left(\circledS_{\jmath}(\wp_1), \circledS_{\jmath}(\wp_1)\right),$$
$$\ell_{\jmath}(\wp_1 \cap \wp_1) \geq \min\left(\ell_{\mathfrak{T}}(\wp_1), \ell_{\jmath}(\wp_1)\right),$$

(iii) $\forall\{\wp_j,\ j \in J\} \subseteq RCI_Z^N$

$$\hbar_{\mathfrak{I}}\left(\bigcup_{j\in J}\wp_j\right) \geq \bigwedge_{j\in J}\hbar_{\mathfrak{I}}(\wp_j),$$

$$Ⓢ_{\mathfrak{I}}\left(\bigcup_{j\in J}\wp_j\right) \geq \bigwedge_{j\in J}Ⓢ_{\mathfrak{I}}(\wp_j),$$

$$\ell_{\mathfrak{I}}\left(\bigcup_{j\in J}\wp_j\right) \geq \bigwedge_{j\in J}\ell_{\mathfrak{I}}(\wp_j).$$

Then we call $\mathfrak{I}$, Riemannian complex gradation of openness and $(Z, \mathfrak{I})$, Riemannian complex fuzzy topological space with Riemannian complex gradation of openness (RCG-fts).

Example 4.6. Let $N = \mathbb{R}^n$ and $Z = \tilde{1}$. we define $\mathfrak{I}_{RCn} : (S^1 \times I)_Z^N \to S^1 \times I$, by

$$\mathfrak{I}_{RCn}(\wp) = \begin{cases} \tilde{1} & \text{if } \wp \in \tau_{_{RCn}}, \\ \tilde{0} & \text{elsewhere.} \end{cases} \tag{4.1}$$

and $\mathfrak{I}_{RC\,\inf} : (S^1 \times I)_Z^N \to S^1 \times I$, by

$$\mathfrak{I}_{RC\,\inf}(\wp) = \left(\hbar_{\mathfrak{I}_{C\,\inf}}(\wp)\ e^{i Ⓢ_{\mathfrak{I}_{C\,\inf}}(\wp)}, \ell_{\mathfrak{I}_{C\,\inf}}(\wp)\right).$$

$$\hbar_{\mathfrak{I}_{RC\,\inf}}(B) = \begin{cases} 1 & \text{if } \wp = \tilde{0}, \\ \inf\{r_B(z) : z \in N\} & if\ \tilde{0} \neq \wp \in \tau_{_{RCn}}, \\ 0 & \text{elsewhere,} \end{cases} \tag{4.2}$$

$$Ⓢ_{\mathfrak{I}_{RC\,\inf}}(\wp) = \begin{cases} 2\pi & \text{if } \wp = \tilde{0}, \\ \inf\{Ⓢ_B(z) : z \in N\} & if\ \tilde{0} \neq \wp \in \tau_{_{RCn}}, \\ 0 & \text{elsewhere,} \end{cases} \tag{4.3}$$

$$\ell_{\mathfrak{I}_{RC\,\inf}}(\wp) = \begin{cases} 1 & \text{if } \wp = \tilde{0}, \\ \inf\{r_B(z) : z \in N\} & \text{if } \tilde{0} \neq \wp \in \tau_{_{RCn}}, \\ 0 & \text{elsewhere,} \end{cases} \tag{4.4}$$

It is clear that both are Riemannian complex gradation of openness. Any Riemannian complex gradation of openness $\mathfrak{I}$ on $\tilde{1}_{\mathbb{R}^n}$, that satisfies the condition $supp\mathfrak{I} = \tau_{_{RCn}}$, then $(\tilde{1}_{\mathbb{R}^n}, \mathfrak{I}_{RCn})$ is called the Riemannian complex fuzzy Euclidean topological space with Riemannian complex gradation of openness.

Definition 4.7. Let $(Z,\mathfrak{I})$ be a RCG-fts. Set $supp\mathfrak{I}=\{\wp\in(S^1\times I)^N_Z:\ \mathfrak{I}(\wp)>0\}$. Then $\wp$ is called an open RCG-fuzzy subset of Z if $\wp\in supp\mathfrak{I}$. Further

1) Let $\wp\in(S^1\times I)^N_Z$ and $z\in Z$ (i.e. $z\in suppZ$). If there exists an open RCG-fuzzzy subset $\wp_1$ of Z, such that $\wp_1(z)=\wp(z)$ and $\wp_1\subset\wp$, then $\wp$ is called a RCG-neighborhood of z in Z. We denote by $RCGN(z)$, the set of all RCG-neighborhoods of z in Z.
2) $(Z,\mathfrak{I})$ is called a Hausdorff RCG-fts, if for all z, $w\in Z$, $z\neq w$, there exist two RCG-neighborhoods $\wp_z\in RCGN(z)$, $\wp_w\in RCGN(w)$, such that $\wp_z\cap\wp_w=\tilde{0}$.
3) For any $\wp\in(S^1\times I)^N_Z$ and each $z\in N$, we define a $RCFS$ on N denoted by $\chi_{\{z\},\wp}$ by
$$\chi_{\{z\},\wp}(t)=\begin{cases}\mu_\wp(z) & t=z,\\ 0 & t\neq z,\end{cases}$$
for all $t\in N$. From now on, we write $\chi_{\{z\}}$ instead of $\chi_{\{z\},Z}$.
4) $\wp$ is called a RCG-closed subset of Z if $Z\backslash\wp\in supp\mathfrak{I}$.

Theorem 4.8. *A RCFS $\wp$ on N is an open RCG-fuzzy subset of Z, if and only if $\wp$ contains a RCG-neighborhood of its any subset.*

Definition 4.9. Let $(Z,\mathfrak{I})$ be a RCG-fts and $\wp\in(S^1\times I)^N_Z$.

1) Let $z\in Z$. If there exists an open RCG-fuzzy subset $\wp_1$ of Z, such that $\mu_{\wp_1}(z)=\mu_\wp(z)$ and $\wp_1\subset\wp$, then $\wp$ is called a RCG-neighborhood, briefly nbd, of z in Z. We denote the set of all RCG-neighborhoods of z in Z by $RCGN(z)$.
2) Let $\wp_1\in(S^1\times I)^N_Z$. Then $\wp_1$ is called an RCG-neighborhood of $\wp$, if there exists an open RCG-fuzzy subset $\wp_2$ such that $\wp\subset\wp_2\subset\wp_1$. We denote the set of all RCG-neighborhoods of $\wp$ by $RCGN(\wp)$.

From now on, we suppose that N_1,N_2 are two crisp sets, Z,Y are two RCFSs on N,N' respectively, and $(Z,\mathfrak{I}),(Y,\mathfrak{L})$ are two RCG-ftss.

Definition 4.10. Let $g:N_1\to N_2$ be a mapping. Then g is called a RCG-related function from $(Z,\mathfrak{I})$ to $(Y,\mathfrak{L})$, if $g[Z]\leq Y$. The set of all these functions is denoted by $RCGRF(Z,Y)$.
Furthermore, g is a gradation-preserving RCG-related function, if $\mathfrak{L}(G)\leq\mathfrak{I}(g^{-1}[G])$ for all open RCG-fuzzy subsets G of Y. So it is called a $RCGP$-related function or $RCGP$-fuzzy mapping from Z to Y, or $g\in RCGPRF(Z,Y)$.

i) If $g|_{suppZ}:\ suppZ\to suppY$ is one-to-one function, then g is called an one-to-one RCG-related function.

ii) If $g[Z] = Y$, then g is called an onto RCG-related function.
iii) If $g[\wp] \in supp\mathfrak{R}$, $\forall \wp \in supp\mathfrak{I}$, then g is called open RCG-related function.
iv) If $g^{-1}[\wp'] \cap Z \in supp\mathfrak{I}$, $\forall \wp' \in supp\mathfrak{L}$, then g is called continuous RCG-related function.
vi) If g is one -to -one, onto, continuous and open, then g is called RCG-homeomorphism.

Proposition 4.11. *Every RCGP-fuzzy mapping from Z to Y is a RCG-continuous related function, but the converse is not true.*

Proposition 4.12. *Let $g : N_1 \to N_2$ be a mapping and $\wp$, $\wp'$ be open RCG-fuzzy subsets of Z, Y respectively. Then g is a RCGP-homeomorphism from $\wp$ to $\wp'$ if and only if g satisfies the following conditions:*

i) $\wp(p) = \wp'(g(p))$ *for any* $p \in \wp$ *or* $\wp'(q) = \wp(g^{-1}(q))$ *for any* $q \in \wp'$,
ii) $\mathfrak{L}(G) = \mathfrak{I}(g^{-1}[G])$ *for any open RCG-fuzzy subset G of $\wp'$.*

Theorem 4.13. *Let $g \in RCGRF(Z, Y)$. If f is a RCG-continuous, then for any $\wp \in (S^1 \times I)_Z^N$ and for any RCG-neighborhood $\wp'$ of $g[\wp]$, there is a RCG-neighborhood $\wp_1$ of $\wp$, such that $g[\wp_1] \subset \wp'$.*

Proof. Let g is a RCG-continuous. Let $\wp \in (S^1 \times I)_Z^N$ and $\wp'$ be a RCG-neighborhood of $g[\wp]$. Then there exists an open RCG-fuzzy subset $\wp_2$, such that $g[\wp] \subset \wp_2 \subset \wp'$.

We have $\wp \subset g^{-1}[g[\wp]] \subset g^{-1}[\wp_2]$, using by (3) and (6) of Theorem 3.7. Set $\wp_1 = g^{-1}[\wp_2] \cap Z$. Since $\wp_2 \in supp\mathfrak{L}$, we have $\wp_1 \in supp\mathfrak{I}$. Thus $\wp \subset \wp_1$ and so $\wp_1$ is a RCG-neighborhood of $\wp$. Thus we have $g[\wp_1] \subset g[g^{-1}[\wp'] \subset \wp'$, using by (4) and (5) of Theorem 3.7

□

Example 4.14. Let $N = \{a_1, a_2\}$ and $N' = \{b_1, b_2\}$. We define two Riemannian complex fuzzy sets Z, Y on N, N' respectively by

$$Z = \{(a_1, 0.8e^{i\pi}, 0.7), (a_2, 0.6e^{i1.1\pi}, 0.8)\}$$
$$Y = \{(b_1, 0.9e^{i1.5\pi}, 0.8), (b_2, 0.9e^{i1.2\pi}, 0.9)\}.$$

and two Riemannian complex gradations of openness $\mathfrak{I}$, $\mathfrak{I}'$ on Z, Y by

$$\mathfrak{I} : (S^1 \times I)_Z^N \to S^1 \times \mathbb{R}, \qquad \mathfrak{I}(\wp) = \begin{cases} (1e^{i2\pi}, 1) & \wp = Z, \tilde{0} \\ (0.4e^{i1.5\pi}, 0.5) & \wp = \wp_1, \\ (0e^{i0}, 0) & \wp \neq \wp_1, \end{cases}$$

$$\mathfrak{I}' : (S^1 \times I)_Z^{N'} \to S^1 \times \mathbb{R}, \qquad \mathfrak{I}'(\wp') = \begin{cases} (1e^{i2\pi}, 1) & \wp' = Y, \tilde{0} \\ (0.6e^{i0.9\pi}, 0.9) & \wp' = \wp_2', \\ (0e^{i0}, 0) & \wp' \neq \wp_2', \end{cases}$$

where

$$\wp_1 = \{(a_1, 0.2e^{i0.3\pi}, 0.7),\ (a_2, 0.6e^{i0.5\pi}, 0.5)\},$$
$$\wp_2' = \{(b_1, 0.9e^{i0.5\pi}, 0.5),\ (b_2,\ 0.2e^{i0.3\pi}, 0.8)\}$$

Hence $supp\mathfrak{I} = \{Z, \tilde{0}, \wp_1\}$ and $supp\mathfrak{L} = \{Y, \tilde{0}, \wp_2'\}$.
Let $g, k : N \to N'$ be two functions defined by $g(z) = b_2, \forall z \in N$ and

$$k(z) = \begin{cases} b_2 & z = a_1, \\ b_1 & z = a_2. \end{cases}$$

We show that $k[Z] \leq Y$:

$$\begin{aligned} k[Z](b_1) &= \sup_{z \in k^{-1}(b_1)} Z(z) \\ &= Z(a_2) \\ &= (0.6e^{i1.1\pi}, 0.8) \\ &\leq (0.9e^{i1.5\pi}, 0.8) \\ &= Y(b_1), \end{aligned}$$

and

$$\begin{aligned} k[Z](b_2) &= \sup_{z \in k^{-1}(b_2)} Z(z) \\ &= Z(a_1) \\ &= (0.8e^{i\pi}, 0.7) \\ &\leq (0.9e^{i1.2\pi}, 0.9) \\ &= Y(b_2). \end{aligned}$$

Also we show that $g[Z] \leq Y$:

$$g^{-1}(b_1) = \phi \quad \Rightarrow \quad g[Z](b_1) = (0e^{i0}, 0) \leq (0.9e^{i1.5\pi}, 0.8) = Y(b_1)$$

$$g[Z](b_2) = \sup_{z \in g^{-1}(b_2)} Z(z) = Z(a_1) \vee Z(a_2) = (0.8e^{i\pi}, 0.7) \vee (0.6e^{i1.1\pi}, 0.8)$$

$$= (0.8e^{i1.1\pi}, 0.8) \leq (0.9e^{i1.2\pi}, 0.9) = Y(b_2)$$

Thus g, k are RCG-related functions from $(Z, \mathfrak{I})$ to $(Y, \mathfrak{L})$.
Now we show that k is RCG-continuous:
For $\tilde{0} \in supp\mathfrak{L}$, we have

$$k^{-1}[\tilde{0}](z) = \tilde{0}(k(z)) = (0e^{i0}, 0) \quad \Rightarrow \quad k^{-1}(\tilde{0}) \cap Z = \tilde{0} \in supp\mathfrak{I}.$$

For $Y \in supp\mathfrak{L}$, we have

$$\begin{aligned} \left(k^{-1}[Y] \cap Z\right)(a_1) &= Y(f(a_1)) \wedge Z(a_1) = Y(b_2) \wedge Z(a_1) \\ &= (0.9e^{i1.2\pi}, 0.9) \wedge (0.8e^{i\pi}, 0.7) = Z(a_1) \end{aligned}$$

$$\begin{aligned}\left(k^{-1}[Y]\cap Z\right)(a_2) &= Y(f(a_2))\wedge Z(a_2) = Y(b_1)\wedge Z(a_2)\\ &= (0.9e^{i1.5\pi},0.8)\wedge(0.6e^{i1.1\pi},0.8) = Z(a_2)\end{aligned}$$

Thus $k^{-1}(Y)\cap Z = Z \in supp\mathfrak{I}$.
For $\wp_2' \in supp\mathfrak{L}$, we see

$$\begin{aligned}(k^{-1}[\wp_2']\cap Z)(a_1) &= \wp_2'(g(a_1))\wedge Z(a_1)\\ &= (0.2e^{i0.3\pi},0.8)\wedge(0.8e^{i\pi},0.7),\\ &= \wp_2'(b_2)\wedge Z(a_1)\\ &= (0.2e^{i0.3\pi},0.7) = \wp_1(a_1)\end{aligned}$$

$$\begin{aligned}(k^{-1}[\wp_2']\cap Z)(a_2) &= \wp_2'(k(a_2))\wedge Z(a_2)\\ &= \wp_2'(b_1)\wedge Z(a_2),\\ &= (0.9e^{i0.5\pi},0.5)\wedge(0.6e^{i1.1\pi},0.8),\\ &= (0.6e^{i0.5\pi},0.5) = \wp_1(a_2).\end{aligned}$$

Thus $k^{-1}[\wp_2']\cap Z = \wp_1 \in supp\mathfrak{I}$. Hence k is RCG-continuous.
We show that g is'nt RCG-continuous:
For $\wp_2' \in supp\mathfrak{L}$, we have

$$\begin{aligned}(g^{-1}[\wp_2']\cap Z)(a_2) &= \wp_2'(g(a_2))\wedge Z(a_2)\\ &= (g^{-1}[\wp_2']\cap Z)(a_2),\\ &= \wp_2'(g(a_2))\wedge Z(a_2),\\ &= \wp_2'(b_2)\wedge Z(a_2),\\ &= (0.2e^{i0.3\pi},0.8)\wedge(0.6e^{i1.1\pi},0.8),\\ &= (0.2e^{i0.3\pi},0.8).\end{aligned}$$

Hence $g^{-1}[\wp_2']\cap Z \notin supp\mathfrak{I}$.

5. Riemannian Complex Fuzzy Topological Manifolds with RC-gradation of Opennes

It is natural that we are led to study spaces that are locally similar to $(1_{\mathbb{R}^n},\mathfrak{I}_{RCn})$, the Riemannian complex fuzzy Euclidean topological space with Riemannian complex gradation of openness, more precisely, spaces for which every point p has an open RCG-neighborhood $\wp$ that

is $RCGP$-homeomorphism to an open RCG-fuzzy subset $\wp'$ of $1_{\mathbb{R}^n}$, with fixed n.

Definition 5.1. Let $\mathfrak{I}$ be a Riemannian complex gradation of openness on Z. If for each $z \in Z$, there exist an open RCG-fuzzy subset $\wp$ of Z containing z and an open RCG-fuzzy subset $\wp'$ of $(1_{\mathbb{R}^n}, \mathfrak{I}_{RCn})$, together a $RCGP$-homeomorphism $\psi \in RCGPRf(\wp, \wp')$, such that $\mathfrak{I}(\wp) = \mathfrak{I}_{RCn}(\wp')$, then $(Z, \mathfrak{I})$ is called a RCG-fuzzy topological space of dimension n.

The pair $(\wp,\ \psi)$ is called a RCG-local coordinate neighborhood of any $q \in \wp$ and we assign to q the n RCG-local coordinates $z_1(t), z_2(t), ..., z_n(t)$, of its image $\psi(t)$ in $\mathbb{R}^n$.

Definition 5.2. A RCG-fuzzy topological space $(Z, \mathfrak{I})$ is called a RCG-fuzzy topological manifold of dimension n, if it satisfies the following conditions:

i) Z is a RCG-fuzzy topological space of dimension n,
ii) Z is a Hausdorff RCG-fts.

Let $\mathfrak{A} = \{(\wp_i, \psi_i)\ i \in J\}$ be a collection of RCG-local coordinate neighborhoods. If $\psi_i \circ \psi_j^{-1}$ and changing RCG-local coordinates $\psi_j \circ \psi_i^{-1}$, are infinitely differentiable or C^∞. Then $(\wp_i, \psi_i)$ is called C^∞ compatible with $(\wp_j, \psi_j)$, whenever $\wp_i \cap \wp_j \neq \phi$.

Definition 5.3. A differentiable or C^∞ RCG-fuzzy structure on a RCG-fuzzy topological manifold $(Z, \mathfrak{I})$, is a family $\mathfrak{A} = \{(\wp_\alpha, \psi_\alpha),\ \alpha \in J\}$ of RCG-local coordinate neighborhoods, such that

1) $Z = \bigcup_{\alpha \in J} \wp_\alpha$,
2) each pair $(\wp_\alpha, \psi_\alpha)$ and $(\wp_\beta, \psi_\beta)$ are compatible for all $\alpha, \beta \in J$,
3) any RCG-local coordinate neighborhood $(\wp, \varphi)$, that is compatible with every $(\wp_\alpha, \psi_\alpha), \alpha \in J$ is in $\mathfrak{A}$ itself.

A RCG-fuzzy topological manifold $(Z, \mathfrak{I})$, with a C^∞ RCG-fuzzy structure on it, is called a C^∞ RCG-fuzzy manifold. For convenience, "RCG-fuzzy manifold with RCG-fuzzy structure" will mean C^∞ RCG-fuzzy manifold with C^∞ RCG-fuzzy structure, in the following of this manuscript.

One reason why C^∞ is a desirable differentiability class to use is that when we later take derivatives functions on manifolds, the assumption of infinite differentiability frees us from many annoying concerns about the order of differentiability.

Example 5.4. Let $N = \mathcal{M}_{m\times n}(\mathbb{R})$. We define $Z \in (S^1 \times I)^N$ by

$$\mu_Z\left((c_{ij})\right) = \left(\hbar_Z\left((c_{ij})\right) e^{i\circledS_Z((c_{ij}))}, \ell_Z\left((c_{ij})\right)\right)$$

$$\hbar_{_Z}\left((c_{ij})\right)=\frac{5}{5+k((c_{ij}))},$$
$$\circledS_{_Z}\left((c_{ij})\right)=2\pi\frac{k((c_{ij}))}{1+k((c_{ij}))},$$
$$\ell_{_Z}\left((c_{ij})\right)=\frac{1}{1+k((c_{ij}))},$$

with $k:N\to\mathbb{N},\quad k((c_{ij}))=max\big\{\lfloor\ |c_{ij}|\ \rfloor\mid 1\leq i\leq m,\ 1\leq j\leq n\big\}$, where $\lfloor z\rfloor$ equals to the greatest integer less than or equal to z. There is an one-to-one mapping:

$$\psi:N\to\mathbb{R}^{mn},\quad \psi\left((c_{ij})\right)=(c_{11},\ldots,c_{1n},\ldots,c_{m1},\ldots,c_{mn})$$

We define

$$\mathfrak{I}_{RCmn}:RCI^{\mathbb{R}^{mn}}_{1_{\mathbb{R}^{mn}}}\to S^1\times I,\qquad \mathfrak{I}_{RCmn}(\wp')=\begin{cases}(1e^{2\pi},1) & \wp'\in\tau_{_{RCmn}},\\ (0e^0,0) & \text{elsewhere,}\end{cases}$$

$$\mathfrak{I}:RCI^{N}_{Z}\to S^1\times I,\qquad \mathfrak{I}(\wp)=\begin{cases}(1e^{2\pi},1) & \psi[\wp]\in\tau_{_{RCmn}},\\ (0e^0,0) & \text{elsewhere.}\end{cases}$$

Let $\wp'$ is a RCG-fuzzy subset of $1_{\mathbb{R}^{mn}}$ defined by

$$\mu_{\wp'_1}(G)=\left(\hbar_{_{\wp'_1}}(G)e^{i\circledS_{\wp'_1}(G)},\ell_{_{\wp'_1}}(G)\right),$$
$$\hbar_{_{\wp'_1}}(G)=\frac{5}{5+(k\circ\psi^{-1})(G)},$$
$$\circledS_{_{\wp'_1}}(G)=2\pi\frac{(k\circ\psi^{-1})(G)}{1+(k\circ\psi^{-1})(G)},$$
$$\ell_{_{\wp'_1}}(G)=\frac{1}{1+(k\circ\psi^{-1})(G)},$$

for all $G=(c_{11},\ldots,c_{1n},\ldots,c_{m1},\ldots,c_{mn})$.
We have $\wp'_1=\psi[Z]$, hence $\psi\in RCGRF(Z,\wp'_1)$. Since $supp\wp'_1=\mathbb{R}^{mn}=\bigcup_{k=1}^{\infty}B(k,k)$, then $\wp'_1\in\tau_{_{RCmn}}$. Thus $\mathfrak{I}_{RCmn}(\wp'_1)=(1e^{i2\pi},1)=\mathfrak{I}(Z)$. So $(Z,\mathfrak{I})$ can be covered by one coodinate neighborhood (Z,ψ). Therefore $(Z,\mathfrak{I})$ is a C^∞ RCG-fuzzy mn-manifold.

6. *RCG*-Fuzzy Quotient Manifolds

In this section, we will discuss the space of Riemannian complex fuzzy (real) projective geometry that arises from equivalence relations defined on simpler Riemannian complex fuzzy manifolds, where the underlying space of the new RCG-fuzzy quotient manifold is a set of equivalence classes with appropriate RCG-fuzzy topology.

Definition 6.1. Suppose that N is a nonempty set and $\sim$ is an equivalence relation on it. If $\wp$ is a RC-fuzzy subset of N, such that $\mu_{\wp}(w) = \mu_{\wp}(z)$ whenever $w \sim z$, then we define the RC-fuzzy subset $\frac{\wp}{\sim}$ by:

$$\mu_{\frac{\wp}{\sim}} : \frac{N}{\sim} \to S^1 \times I, \qquad \mu_{\frac{\wp}{\sim}}([z]) = \mu_{\wp}(z), \quad \forall z \in N,$$

where $[z] = \{w \mid z \sim w\}$. Since $\wp \leq Z$, hence $\frac{\wp}{\sim} \leq \frac{Z}{\sim}$ and so

$$\frac{\wp}{\sim} \in (S^1 \times I)^{\frac{M}{\sim}}_{\frac{Z}{\sim}}.$$

Theorem 6.2. *Assume that $(Z, \Im)$ is a RCG-fuzzy topological space, such that $\mu_{Z}(w) = \mu_{Z}(z)$ whenever $w \sim z$, then $\left(\frac{Z}{\sim}, \frac{\Im}{\sim}\right)$ is a RCG-fuzzy topological space, called the RCG-fuzzy quotient space, where*

$$\frac{\Im}{\sim} : (S^1 \times I)^{\frac{N}{\sim}}_{\frac{Z}{\sim}} \to S^1 \times I, \qquad \frac{\Im}{\sim}(\frac{\wp}{\sim}) = \Im(\wp).$$

Since we have for each $\alpha \in I$, $\mu_{\alpha Z}(z) = \mu_{\alpha Z}(w)$ whenever $z \sim w$. Hence $\alpha Z \in supp\frac{\Im}{\sim}$. Therefore this RCG-fuzzy topology is nontrivial.

Definition 6.3. Suppose that $\left(\frac{Z}{\sim}, \frac{\Im}{\sim}\right)$ be a RCG-fuzzy quotient space. The equivalence relation $\sim$ is called an open RCG-relation if for any fuzzy subset $\wp \in supp\Im$, we have $\frac{\wp}{\sim} \in supp\frac{\Im}{\sim}$.

Theorem 6.4. *If $(Z, \Im)$ is a RCG-fuzzy manifold and $\sim$ is an open RCG-relation. Then $\left(\frac{Z}{\sim}, \frac{\Im}{\sim}\right)$ is a RCG-fuzzy topological space of dimension n, called RCG-fuzzy quotient topological space.*

Example 6.5. We define a relation on $\mathbb{R}$ by:

$$\forall z, w \in 1_{\mathbb{R}}, \qquad z \sim w \quad \text{if } z - w \in \mathbb{Z}.$$

Since $1_{\mathbb{R}}(w) = 1_{\mathbb{R}}(z) = (1e^{i2\pi}, 1)$, whenever $w \sim z$, hence $\frac{1_{\mathbb{R}}}{\sim}$ is well-defined and $\left(\frac{1_{\mathbb{R}}}{\sim}, \frac{\Im_{RC1}}{\sim}\right)$, is a RCG-fuzzy quotient topological space using by Theorem 6.2. Also it is a RCG-fuzzy topological manifold as we show in the following:
Let $\mu_{\wp} : \mathbb{R} \to S^1 \times I$, $\mu_{\wp}(z) = \left(\hbar_{\wp}(z)e^{i\circledS_{\wp}(z)}, \ell_{\wp}(z)\right)$ with

$$\hbar_{\wp}(z) = z - \lfloor z \rfloor, \qquad \circledS_{\wp}(z) = 2(z - \lfloor z \rfloor)\pi, \qquad \ell(z) = \frac{z - \lfloor z \rfloor}{6}$$

Since

$$supp\wp = \mathbb{R} - \mathbb{Z} = \bigcup_{k\in\mathbb{Z}}(k,\ k+1) = \bigcup_{k\in\mathbb{Z}} B\left(k+\frac{1}{2},\ \frac{1}{2}\right),$$

then we see

$$\wp = \bigcup_{k\in\mathbb{Z}} B\left(k+\frac{1}{2},\frac{1}{2},b_k,c_k,d_k\right)$$

with

$$\mu_{B\left(k+\frac{1}{2},\frac{1}{2},b_k,c_k,d_k\right)}(z) = \left(b(z)e^{ic(z)},d(z)\right),$$

where

$$b_k(z) = \begin{cases} z-\lfloor z\rfloor & z\in B\left(k+\frac{1}{2},\ \frac{1}{2}\right), \\ 0 & \text{otherwise,}\end{cases}$$

$$c_k(z) = \begin{cases} 2(z-\lfloor z\rfloor)\pi & z\in B\left(k+\frac{1}{2},\ \frac{1}{2}\right), \\ 0 & \text{otherwise,}\end{cases}$$

$$d_k(z) = \begin{cases} \dfrac{z-\lfloor z\rfloor}{6} & z\in B\left(k+\frac{1}{2},\ \frac{1}{2}\right), \\ 0 & \text{otherwise.}\end{cases}$$

We have $\wp\in\tau_{RC1}$ using by Example 4.2 and we have $\Im_{RC1}(\wp)=1$ using by Example 4.6.

7. C^∞ *RCG*-Fuzzy Mappings of *RCG*-Fuzzy Manifolds

Definition 7.1. Suppose that $(Z,\Im)$ is a *RCG*-fuzzy n-manifold, $\wp\in supp\Im$ and $\wp'\in supp\Im_{RC1}$. Then *RCG*-related function $f\in RCGRF(\wp,\wp')$ is called a C^∞ *RCG*-fuzzy mapping, if for every $p\in\wp$,

$$\hat{g} = g\circ\psi_1^{-1} : \psi_1(supp(\wp_1\cap\wp)) \ \to\ supp\wp'$$

is C^∞, where $(\wp_1,\psi_1)$ is a *RCG*-local coordinate neighborhood of p.

We denote by $C^\infty_{RC}(p)$, the set of all C^∞ *RCG*-fuzzy mappings from an open *RCG*-fuzzy subset $\wp$ of Z, containing p to $\mathbb{R}$. Assume that $\psi_1(t)=(z_1,\ldots,z_n)$, $\forall t\in supp(\wp_1\cap\wp)$. Hence $\hat{g}(z_1,\ldots,z_n)=w(t)$ and since $\hat{g}$ is C^∞, then all partial derivatives of any order of w exist.

From now on, we assume that $N_1,\ N_2$ are two crisp sets, $Z\in(S^1\times I)^{N_1}, Y\in(S^1\times I)^{N_2}$, such that $(Z,\Im)$ is a *RCG*-fuzzy n-manifold with *RCG*-fuzzy structure $\mathfrak{A}=\{(\wp_i,\psi_i), i\in K\}$ and $(Y,\mathfrak{L})$ is a *RCG*-fuzzy m-manifold with *RCG*-fuzzy structure $\mathfrak{D}=\{(\wp'_j,\varphi_j),\ j\in J\}$ and also $\wp\in supp\Im$.

Example 7.2. If we define $g : \mathcal{M}_{m\times n} \to \mathbb{R}$ by $g((c_{ij})) = det(c_{ij})$ in example 5.4, and consider RCG-local coordinate neighborhood $(Z, \psi|_Z)$, then $\hat{g} = g \circ \psi$ is C^∞. Thus $g \in C^\infty_{RC}(p)$ for all $p \in Z$.

Definition 7.3. A RC-related function $g = RCGRF(\wp, Y)$ is a C^∞ RCG-fuzzy mapping if for any $p \in \wp_1$,

$$\hat{g} = \varphi \circ g \circ \psi^{-1} : \psi(supp(\wp_1 \cap \wp)) \ \to \ \varphi(supp\ \wp_1')$$

is C^∞, where $(\wp_1, \psi)$, $(\wp_1', \varphi)$ are RCG-local coordinate neighborhoods of p and $g(p)$ respectively.
Also $g = RCGRF(Z, Y)$ is called a RCG-diffeomorphism if it is a RCG-homeomorphism and g^{-1} is C^∞.

If $\psi(t) = (z_1, \dots, z_n)$, $\forall t \in supp(\wp \cap \wp_1)$ and $\varphi(w) = (w_1, \dots, w_m)$, $\forall w \in \wp_1'$ we have

$$\hat{g}(z_1, \dots, z_n) = (g_1(z_1, \dots, z_n), \dots, g_n(z_1, \dots, z_n))$$

and each $w_i = g_i(z_1, \dots, z_n)$ is C^∞ on $\psi(\wp)$.

Definition 7.4. The rank of the Jacobian matrix

$$\begin{pmatrix} \frac{\partial g_1}{\partial z_1} & \cdots & \frac{\partial g_1}{\partial z_n} \\ \vdots & & \vdots \\ \frac{\partial g_m}{\partial z_1} & \cdots & \frac{\partial g_m}{\partial z_n} \end{pmatrix}_z$$

at $z = \psi(t)$, is called the rank of $g \in LRf(Z, Y)$ at p.

Example 7.5. Let $N_1 = N_2 = \mathbb{R}^2$ and $Z : N_1 \to S^1 \times I$ be defined by $\mu_{_Z}(z) = \left(\hbar_{_Z}(z)e^{i\circledS_Z(z)}, \ell_{_Z}(z)\right)$ for all $z = (z_1, z_2)$ where

$$\hbar_{_Z}(z) = \begin{cases} |z_1 - z_2| & \|z\| = 1, \\ 0 & \|z\| \neq 1, \end{cases},$$

$$\circledS_{_Z}(z) = \begin{cases} \frac{\pi}{4} & \|z\| = 1, \\ 0 & \|z\| \neq 1, \end{cases}$$

$$\ell_{_z}(z) = \begin{cases} \sqrt{z_1 z_2} & \|z\| = 1, \\ 0 & \|z\| \neq 1. \end{cases}$$

and $Y : N_2 \to S^1 \times I$ be defined by $\mu_{_Y}(w) = \Big(\hbar_{_Y}(w)e^{i\circledS_{_Y}(w)}, \ell_{_Y}(w)\Big)$, for all $w = (w_1, w_2)$, where

$$\hbar_{_Y}(w) = \begin{cases} |w_1| & \|w\| = 1, \\ 0 & \|w\| \neq 1, \end{cases},$$

$$\circledS_{_Y}(w) = \begin{cases} \dfrac{\pi}{4} & \|w\| = 1, \\ 0 & \|w\| \neq 1, \end{cases}$$

$$\ell_{_Y}(w) = \begin{cases} \dfrac{|w_2|}{\sqrt{2}} & \|w\| = 1, \\ 0 & \|w\| \neq 1. \end{cases}$$

We define $\mathfrak{I} : (S^1 \times I)_Z^{N_1} \to S^1 \times I, and\ \mathfrak{L} : (S^1 \times I)_Y^{N_2} \to (S^1 \times I)$, by

$$\mathfrak{I}(A) = \begin{cases} 1 & A \in supp\mathfrak{I}_{RC3}, A \leq Z, \\ 0 & \text{elsewhere.} \end{cases}$$

$$\mathfrak{L}(D) = \begin{cases} 1 & D \in supp\mathfrak{I}_{RC3}, D \leq Y, \\ 0 & \text{elsewhere.} \end{cases}$$

Then $(Z, \mathfrak{I}), (Y, \mathfrak{L})$ are RCG-fuzzy manifolds, using by Example 4.6. Let

$$g : N_1 \to N_2,\ g(z) = (z_1 - z_2, \sqrt{2z_1z_2}).$$

Then

$$z \in S^1 \ \Rightarrow\ z_1^2 + z_2^2 = 1 \ \Rightarrow\ (z_1 - z_2)^2 + (\sqrt{2z_1z_2})^2 = 1 \ \Rightarrow\ g(z) \in S^1,$$

Hence $g|_{suppZ} : suppZ \to suppY$ is well-defined function. Now we show that $g[Z] = Y$:

$$\begin{aligned} \hbar_{g[Z]}(w) &= \sup\big\{\hbar_Z(z) :\ z \in g^{-1}(w)\big\} = \sup\big\{|z_1 - z_2| :\ z \in g^{-1}(w)\big\} \\ &= |w_1| = \hbar_Y(w). \\ \circledS_{g[Z]}(w) &= \sup\big\{\circledS_Z(z) :\ z \in g^{-1}(w)\big\} = \sup\big\{\frac{\pi}{4} :\ z \in g^{-1}(w)\big\} \\ &= \frac{\pi}{4} = \circledS_Y(w). \\ \ell_{g[Z]}(w) &= \sup\big\{\ell_Z(z) :\ z \in g^{-1}(w)\big\} = \sup\big\{\sqrt{z_1z_2} :\ z \in g^{-1}(w)\big\} \\ &= \frac{|w_2|}{\sqrt{2}} = \ell_Y(w). \end{aligned}$$

Therefore $g \in RCGRF(Z, Y)$ is a C^∞ RCG-fuzzy mapping

Theorem 7.6. *Let $(Z, \mathfrak{I})$ be a RCG-fuzzy n-manifold with RCG-fuzzy structure $\mathfrak{A} = \{(\wp_j, \psi_j), j \in J\}$.*

If $\mathfrak{I}^{\triangleleft} = \{supp\ \wp \mid \wp \in (S^1 \times I)_Z^{N_1}\}$, then $(suppZ, \mathfrak{I}^{\triangleleft})$ is a n- dimensional manifold called a premanifold with the structure

$$\mathfrak{A}^{\triangleleft} = \{(supp\wp_j, \psi_j|_{supp\wp_j}), j \in J\},$$

called a prestructure.

Theorem 7.7. *(RCG-fuzzy rank theorem) Let $g \in RCGRF(U, Y)$ be a C^∞ fuzzy mapping and* $\operatorname{rank} g = k$ *at each point of Z. If $p \in Z$, then there exist RCG-local coordinate neighborhoods $(\wp, \psi)$, $(\wp', \varphi)$, such that*

$$\psi(p) = (0, \dots, 0) \in \mathbb{R}^n,\ \varphi(g(p)) = (0, \dots, 0) \in \mathbb{R}^m \tag{7.1}$$

and $\hat{g} = \varphi \circ g \circ \psi^{-1}$ is defined by:

$$\hat{g}(z_1, \dots, z_n) = (z_1, \dots, z_k, 0, \dots, 0) \tag{7.2}$$

Remark 7.8. *RCG*-fuzzy n-manifolds Z and $\tilde{Z} = g[Z]$ can be covered by these *RCG*-local coordinate neighborhoods $\mathfrak{A} = \{(\wp_s, \psi_s)|\ s \in S\}$, and $\mathfrak{D} = \{(\wp'_s, \varphi_s)|\ s \in S\}$ respectively. Since $\mathfrak{A}$ is a *RCG*-structure of Z, we can show that $\mathfrak{D}$ is a *RCG*-structure of $\tilde{Z}$.

If g is a *RCG*-diffeomorphism, then $\operatorname{rank} g = \dim Z = \dim Y$.

Definition 7.9. The C^∞ *RCG*-fuzzy function $g \in RCGRF(Z, Y)$ is called a *RCG*-fuzzy immersion (*RCG*-fuzzy submersion), if $\operatorname{rank} g = \dim Z\ (= \dim Y)$ at each point of Z.

Theorem 7.10. *Suppose that $g \in RCGRF(Z, Y)$ is a C^∞ RCG-fuzzy mapping . Set $\tilde{Z} = g[Z]$. If F is an one-to-one RCG-fuzzy immersion, then $(\tilde{Z}, \tilde{\mathfrak{I}})$ is a RCG-fuzzy n-submanifold, called a RCG-fuzzy immersed submanifold and $g \in RCGRF(Z, \tilde{Z})$ is a RCG-diffeomorphism.*

Proof. g is a bijection from $suppZ$ to $g(suppZ)$. Hence, $g \in RCGRF(Z, \tilde{Z})$. Since for any $t \in g(suppZ)$, there is an uinque $b \in suppZ$, such that $g^{-1}(t) = \{b\}$. So

$$\tilde{Z}(t) = g[Z](t) = sup\{Z(a)|\ g(a) = t\} = Z(b).$$

Since $g \in RCGRF(Z, Y)$, then $g[Z] \leq Y$; thus $\tilde{Z} \leq Y$. So $\tilde{Z}$ is a *RCG*-fuzzy subset of Y. We use g to assign $\tilde{Z}$ a *RCG*-fuzzy gradation of openness

$$\tilde{\mathfrak{I}} : (S^1 \times I)_{\tilde{Z}}^{N_2} \to S^1 \times I, \qquad \tilde{\mathfrak{I}}(\wp') = \mathfrak{I}(g^{-1}[\wp']).$$

Therefore for every $p \in Z$, using by Theorem 7.7, there exist *RCG*-local coordinate neighborhoods $(\wp, \psi)$, $(\wp', \varphi)$ of p and $g(p)$ such that

$$(\varphi \circ g \circ \psi^{-1})(z_1, \dots, z_n) = (z_1, \dots, z_n, 0, \dots, 0).$$

Then $(\pi \circ \varphi \circ g)(z) = \psi(z)$ that $\pi(w_1, \ldots, w_m) = (w_1, \ldots, w_n)$ is the projection map.
For each $t \in g(supp\wp)$, there exists only one $b \in supp\wp$ so that $g(b) = t$. Thus we have

$$(\pi \circ \varphi)(t) = (\pi \circ \varphi \circ g)(b) = \psi(b) \leq Z(b) = \tilde{Z}(t).$$

So $\pi \circ \varphi \leq \tilde{Z}$. Hence

$$\tilde{\mathfrak{D}} = \{(\wp'_s, \pi \circ \varphi_s) \mid (\wp'_s, \varphi_s) \in \mathfrak{D},\ \forall s \in S\}$$

is a RCG-fuzzy structure of $\tilde{Z}$. Therefore $(\tilde{Z}, \tilde{\mathfrak{I}})$ is a RCG-fuzzy n-submanifold, called a RCG-fuzzy immersed submanifold and $g \in RCGRF(Z, \tilde{Z})$ is a RCG-diffeomorphism. □

Since RCG-fuzzy gradation of openness $\tilde{\mathfrak{I}}$ and the RCG-fuzzy structure on $\tilde{Z}$ depend on g and Z, we can not say that $(\tilde{Z}, \tilde{\mathfrak{I}})$ is a RCG-fuzzy submanifold of $(Y, \mathfrak{L})$. We need some conditions which are given in the following definition:

Definition 7.11. A RCG-fuzzy imbedding is an one-to-one RCG-fuzzy immersion $g \in RCGRF(Z, Y)$ such that $g \in RCGRF(Z, \tilde{Z})$ is a RCG-homeomorphism from Z to $\tilde{Z} = g[Z]$ as a RCG-fuzzy subspace of Y. The image of a RCG-fuzzy imbedding is called a RCG-fuzzy imbedded submanifold.

It means that g is a RCG-homeomorphism from $(Z, \mathfrak{I})$ to $(\tilde{Z}, \mathfrak{L}|_{\tilde{Z}})$. Hence for all open RCG-fuzzy subsets $\wp$ of Z, we have $F(\wp) = W \cap \tilde{Z}$ where W is an open RCG-fuzzy subset of Y. Also we see

$$\tilde{\mathfrak{I}}(G) = \mathfrak{I}(g^{-1}[G]) = \mathfrak{L}(G),$$

for all open RCG-fuzzy subsets G of $\tilde{Z}$.

Notice an interesting difference between ordinary topological manifolds and Riemannian complex fuzzy topological manifolds in the following Remark.

Remark 7.12. Let $g \in RCGRF(Z, Y)$ be a RCG-fuzzy immersion. Then for each $p \in Z$, dosn't necesarily exist a RCG-neighborhood $\wp$ of p so that $g|_{supp\wp}$ is a RCG-fuzzy imbedding.

Proof. It is enough to give an example. Let $N_1 = \mathbb{R}$, $N_2 = \mathbb{R}^2$ and Z, Y be two Riemannian complex fuzzy sets with MFs, $\mu_Z(z) = \left(\frac{1}{5}e^{i2\pi}, \frac{3}{\pi}\right)$ and $\mu_Y(z) = \left(\frac{1}{2}e^{i2\pi}, \frac{6}{\pi}\right)$ respectively.
Let $\mathfrak{I} : (S^1 \times I)^{N_1}_Z \to S^1 \times I$, be defined by

$$\mathfrak{I}(\wp) = \begin{cases} \left(\frac{1}{3}e^{i2\pi}, \frac{1}{4}\right) & \text{if } \wp \in \tau_{RC1}, \\ \tilde{0} & \text{elsewhere.} \end{cases} \tag{7.3}$$

and $\mathfrak{L} : (S^1 \times I)^{N_2}_Y \to S^1 \times I$, be defined by

$$\mathfrak{L}(\wp') = \begin{cases} (\frac{1}{2}e^{i2\pi}, 1) & \text{if } \wp' \in \tau_{RC2}, \\ \tilde{0} & \text{elsewhere.} \end{cases} \tag{7.4}$$

Then $(Z, \mathfrak{I})$ and $(Y, \mathfrak{L})$ are two RCG-fuzzy topological manifolds. If $g : N_1 \to N_2$ is defined by $g(z) = (z, 0)$. Then $g[Z] \leq Y$ and $rankg = dimZ = 1$.

Hence g is a RCG-fuzzy immersion. Since for each open RCG-fuzzy subset G of Y,

$$\mathfrak{I}(F^{-1}[G]) = \left(\frac{1}{3}e^{i2\pi}, \frac{1}{4}\right) \neq \mathfrak{L}(G),$$

then, $g|_{supp\wp}$ isn't a RCG-fuzzy imbedding for all open RCG-fuzzy subsets $\wp$ of Z. □

8. RCG-Fuzzy Submanifolds of RCG-Fuzzy Manifolds

Definition 8.1. Let X be a RCG-fuzzy subset of a RCG-fuzzy manifold $(Z, \mathfrak{I})$. If any $p \in Z$ has a RCG-local coordinate neighborhood $(\wp, \psi)$ on Z with RCG-local coordinates $z_1, z_2, \ldots, z_n$, such that $\psi(p) = (0, \ldots, 0) \in \mathbb{R}^n$, and

$$\psi(supp\wp \cap suppZ) = \{(z_1, z_2, \ldots, z_n) \in \psi(\wp) \mid z_{k+1} = \ldots = z_n = 0\}.$$

Then X is said to have the RCG-fuzzy k-submanifold property and these type RCG-coordinate neighborhoods, are called preferred RCG-local coordinates.

Using the notation $\pi : \mathbb{R}^n \to \mathbb{R}^k,\ k \leq n$, the projection to the first k coordinates, we prove the following theorem:

Theorem 8.2. *Let $X \subseteq Z$ has the RCG-fuzzy k-submanifold property. Then every preferred RCG-local coordinate system $(\wp, \psi)$ of Z defines a RCG-local coordinate neighborhood $(\wp', \psi')$ on N, that $\wp' = \wp \cap X$, $\psi' = \pi \circ \psi|_{\wp'}$. Thus, the inclusion $i \in RCRf(X, Z)$ is a RCG-fuzzy imbedding.*

Proof. Assume that X has RCG-fuzzy relative topology as a subspace of Z. So $(\wp', \psi')$ are RCG-coordinate neighborhoods covering X, that $\wp' = \wp \cap Z$ is an open RCG-fuzzy subset of X in the relative gradation of openness and $\psi' = \pi \circ \psi|_{\wp'}$ is a RCG-homeomorphism.

Consider $(\wp'_1, \psi'_1)$ and $(\wp'_2, \psi'_2)$ such that $\wp'_1 \cap \wp'_2 \neq \emptyset$. We show that $\psi'_1 \circ \psi'^{-1}_2$ and $\psi'_2 \circ \psi'^{-1}_1$, are C^∞. Let

$$\gamma(z_1, \ldots, z_k) = (z_1, \ldots, z_k, 0, \ldots, 0) \in \mathbb{R}^n.$$

For two preferred neighborhoods, such that $\pi \circ \gamma$ is identity on $\mathbb{R}^k$, this map γ is C^∞. Thus its restriction on $\psi'(\wp')$, an open RCG-fuzzy subset of $\mathbb{R}^k$, is C^∞. Since $\psi'^{-1} = \psi \circ \gamma$ is a composition of C^∞ maps, ψ'^{-1} is

C^∞. Also we see, $\psi' = \pi \circ \psi$. Hence ψ' is a C^∞ map on $\wp'$.
If $w_i = f_i(z_1, \ldots, z_k), i = 1, ..., k$, are the functions giving $\psi_1' \circ \psi_2^{'-1}$, that are C^∞, then, it is seen that $\psi_1 \circ \psi_2^{-1}$ given by $w_i = f_i(z_1, \ldots, z_k, 0, \ldots, 0)$, $i = 1, ..., k$. Therefore $\psi_1' \circ \psi_2^{'-1}$ is C^∞. Therefore these RCG-neighborhoods define a unique differentiable structure on X. In the preferred RCG-local coordinates, $(\wp', \psi') \in RCRf(X, Z)$ is defined by

$$\psi \circ i \circ \psi^{'-1}(z_1, \ldots, z_k) = (z_1, \ldots, z_k, 0, \ldots, 0).$$

Hence the map i is a RCG-fuzzy immersion.

Taking the relative RCG-fuzzy topology on Z, the fuzzy map i is a RCG-homeomorphism to its image $i(X)$, with the RCG-fuzzy subspace topology. Hence i is a RCG-fuzzy imbedding. □

Definition 8.3. A regular RCG-fuzzy submanifold of a RCG-fuzzy manifold $(Z, \mathfrak{I})$ is any RCG-fuzzy topological subspace Z with the RCG-fuzzy submanifold property and with the structure, determined by the corresponding preferred RCG-local coordinate neighborhoods.

Remark 8.4. If $X \subseteq Z$ is a RCG-fuzzy regular submanifold. Since the inclusion $i : X \to N$ is a RCG-fuzzy imbedding as we proved in Theorem 8.2, then X is also a RCG-fuzzy imbedded submanifold.

Theorem 8.5. *Let $g \in RCRf(Z', Z)$ be a RCG-fuzzy imbedding of a RCG-fuzzy k-manifold Z' in a RCG-fuzzy n-manifold. Then $Z = g[Z']$ has RCG-fuzzy k-submanifold property and hence Z is a RCG-fuzzy regular submanifold.*

Also using $g \in RCRF(Z', Z)$, we have Z is diffeomorphic to Z' .

Proof. Let $t = g(p)$ be an element of Z. Using by Theorem 7.7, there exist $(\wp, \psi)$ and $(\wp', \varphi)$, RCG-local coordinate neighborhoods of p and $g(p)$ respectively, satisfying (7.1) and (7.2).

If $g[\wp] = U \subseteq Z$, then the RCG-neighborhood U would be a preferred RCG-local coordinate neighborhood relative to Z. Since g is a RCG-fuzzy imbedding, then $g[\wp]$ is a relatively open RCG-fuzzy subset of Z, that is, $g[\wp] = E \cap Z$, where E is open RCG-fuzzy subset of Z and $\varphi[g[\wp]] \subseteq \varphi[E]$, which $\forall z \in supp\varphi[U]$, $z_{k+l} = \ldots = z_m = 0$.

Thus we can choose an (smaller) open RCG-fuzzy subset $\varphi[U_1] \subseteq \varphi[E]$ and $\varphi_1 = \varphi|_{U_1}$. This is a RCG-local coordinate neighborhood of t for which $g[\wp] \cap U_1 = U_1 \cap Z$.

Furthermore, taking $\wp_1 = g^{-1}(U_1)$, we have $(\wp_1, \psi_1)$, with $\psi_1 = \psi|_{\wp_1}$, is a RCG-local coordinate neighborhood of p and the pair $(\wp_1, \psi_1)$ and (U_1, φ_1) satisfies (7.1), (7.2) and $g[\wp_1] = U_1 \subseteq Z$.

Since the inverse of $g \in RCRf(Z', Z)$ is given in the preferred RCG-local coordinates $(U_1, \pi \circ \varphi_1)$ and $(\wp_1, \psi_1)$ by $\hat{g}^{-l}(z_l, \ldots, z_k) = (z_1, \ldots, z_k)$, which is C^∞, hence Z is a RCG-fuzzy k-submanifold. □

Example 8.6. Consider the map $g : \mathbb{R}^3 \to \mathbb{R}$ given by

$$g(z^1, z^2, z^3) = \left(a - \sqrt{(z^1)^2 + (z^2)^2}\ \right)^2 + (z^3)^2.$$

It can be shown that g is a RCG-related function between two the Riemannian complex fuzzy Euclidean topological spaces $(\tilde{1}_{\mathbb{R}^3}, \Im_{RC3})$ and $(\tilde{1}_{\mathbb{R}^1}, \Im_{RC1})$. Also g is a RC-gradation-preserving RCG-related function and has rank 1 at every point of $g^{-1}(b^2),\ 0 < b < a$. So g is a RCG-fuzzy submersion, since $rankg = dim\tilde{1}_{\mathbb{R}^1}$ at every point of $\tilde{1}_{\mathbb{R}^3}$. Thus $g^{-1}(b^2)$, the torus in $\mathbb{R}^3$ is a submanifold.

9. Conclusion

In this paper, we define the image and inverse image of a Riemannian complex fuzzy subspace under a function. Then we introduce Riemannian complex fuzzy topological space (n-manifold) $(Z, \Im)$ with Riemannian complex gradation of openness, which Z is itself a Riemannian complex fuzzy subset of a nonempty set N $(RCGfts)$. Then we discuss various functions between two $RCGftss$. Finally, we define imbedded and regular RCG-fuzzy submanifolds and then Some theorems are deduced about the relationships between them.

In the continuation of our research, two question arise, how can this model be extended to Riemannian complex fuzzy finsler manifolds or Riemannian complex fuzzy vector bundles?

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